Journal of Number Theory 130 (2010) 2480–2527

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the lifting of elliptic cusp forms to cusp forms on quaternionic unitary groups

Shunsuke Yamana [1]

Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

ARTICLE INFO

Article history:
Received 15 February 2008
Revised 25 July 2009
Available online 8 July 2010
Communicated by Stephen Rallis

MSC:
11F55
11F30

Keywords:
Quaternion algebras
Ikeda lifting
Maass lifting

ABSTRACT

Let H be a definite quaternion algebra over $\mathbb{Q}$ with discriminant D_H and R a maximal order of H. We denote by G_n a quaternionic unitary group and put $\Gamma_n = G_n(\mathbb{Q}) \cap \mathrm{GL}_{2n}(R)$. Let $S_\kappa(\Gamma_n)$ be the space of cusp forms of weight κ with respect to Γ_n on the quaternion half-space of degree n. We construct a lifting from primitive forms in $S_k(\mathrm{SL}_2(\mathbb{Z}))$ to $S_{k+2n-2}(\Gamma_n)$ and a lifting from primitive forms in $S_k(\Gamma_0(d))$ to $S_{k+2}(\Gamma_2)$, where d is a factor of D_H. These liftings are generalizations of the Maass lifting investigated by Krieg.

To my father

0. Introduction

The purpose of this paper is to construct a lifting that associates to an elliptic cusp form a cusp form on a quaternionic unitary group. This is a quaternionic modular analogue of the liftings constructed by Ikeda [14,15]. In a similar fashion, Ikeda constructed, from an elliptic cusp form, a Siegel cusp form in [14] and a hermitian cusp form in [15].

Let us describe our results. Let H be a definite quaternion algebra over $\mathbb{Q}$ and $^\iota$ the main involution of H. Fix a maximal order R of H. Let $\mathbb{H} = H \otimes_\mathbb{Q} \mathbb{R}$, $H_p = H \otimes_\mathbb{Q} \mathbb{Q}_p$ and $R_p = R \otimes_\mathbb{Z} \mathbb{Z}_p$. Put $x^* = {}^t x^\iota$ for $x \in \mathrm{M}_n(H)$.

E-mail address: yamana@sci.osaka-cu.ac.jp.

[1] We would like to thank Prof. Ikeda for suggesting this problem and for his valuable advice and guidance. Parts of this work were done while the author was visiting Aachen University. The author thanks the staffs of Aachen University, especially Prof. Krieg, for their hospitality and encouragement. The author is supported by the Grant-in-Aid for JSPS Fellows.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.03.020

Let G_n be a connected algebraic group defined over $\mathbb{Q}$ whose group of $\mathbb{Q}$-valued points is given by

$$G_n(\mathbb{Q}) = \left\{ \alpha \in \mathrm{SL}_{2n}(H) \;\middle|\; \alpha \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \alpha^* = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \right\}.$$

The modular group is defined to be $\Gamma_n = \mathrm{GL}_{2n}(R) \cap G_n(\mathbb{Q})$.

For a ring $\mathcal{O}$ with involution ${}^\iota$, we put $S_n(\mathcal{O}) = \{x \in \mathrm{M}_n(\mathcal{O}) \mid {}^t x^\iota = x\}$. The quaternion upper half-space of degree n is defined by

$$\mathfrak{H}_n = \left\{ Z = X + \sqrt{-1}Y \in S_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \;\middle|\; X \in S_n(\mathbb{H}),\ 0 < Y \in S_n(\mathbb{H}) \right\}.$$

For any $\mathbb{Q}$-algebra D, let ν, $\tau : \mathrm{M}_n(H \otimes_{\mathbb{Q}} D) \to D$ be the reduced norm and the reduced trace on $\mathrm{M}_n(H \otimes_{\mathbb{Q}} D)$ respectively. Put $\lambda = \frac{1}{2}\tau$. We define a polynomial map $\mathrm{Paf} : S_n(H) \to \mathbb{Q}$, using the relations

$$\mathrm{Paf}(\mathbf{1}_n) = 1, \qquad \mathrm{Paf}(X)^2 = \nu(X), \quad X \in S_n(H).$$

Let κ be an even integer. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_n(\mathbb{R})$, $Z \in \mathfrak{H}_n$ and a function F on $\mathfrak{H}_n$, we put

$$\alpha Z = (aZ + b)(cZ + d)^{-1}, \qquad F|_\kappa \alpha(Z) = \nu(cZ + d)^{-\kappa/2} F(\alpha Z).$$

When $n \geqslant 2$, a modular form F of weight κ is a holomorphic function on $\mathfrak{H}_n$ which satisfies $F|_\kappa \gamma = F$ for every $\gamma \in \Gamma_n$. Put

$$T_n = \left\{ h \in S_n(H) \;\middle|\; \lambda(h\beta) \in \mathbb{Z} \text{ for every } \beta \in S_n(R) \right\}$$

and let T_n^+ denote the set of positive definite elements of T_n. A modular form F is called a cusp form if it has a Fourier expansion of the form

$$F(Z) = \sum_{h \in T_n^+} A_F(h) \mathbf{e}\big(\lambda(hZ)\big)$$

(cf. Remark 1.3). Let $S_\kappa(\Gamma_n)$ be the space of cusp forms on $\mathfrak{H}_n$ of weight κ.

Krieg systematically developed the theory of modular forms on $\mathfrak{H}_n$ in [22], but he makes the following assumptions:

(I) H is the Hurwitz quaternion, i.e., H has a basis $\{1, i, j, k\}$ over $\mathbb{Q}$ such that $k = ij = -ji$, $i^2 = j^2 = -1$;
(II) R is the Hurwitz order, i.e., $R = \mathbb{Z}[i, j, k, \frac{1+i+j+k}{2}]$.

The present paper investigates modular forms with respect to the group Γ_n which comes from an arbitrary definite quaternion algebra over $\mathbb{Q}$.

Fix a rational prime p. The Siegel series attached to $h \in T_n$ is defined by

$$b_p(h, s) = \sum_{\beta \in S_n(H_p)/S_n(R_p)} \mathbf{e}_p\big(-\lambda(h\beta)\big) \nu[\beta]^{-s/2},$$

where $\nu[\beta] = [\beta R_p^n + R_p^n : R_p^n]^{1/2}$.

Let D_H be the discriminant of H. Put $D_h = D_H^{[n/2]} \mathrm{Paf}\, h$. If h is nondegenerate, then there exists a polynomial $F_{p,h}$ such that

$$F_{p,h}\bigl(p^{-s}\bigr)=b_p(h,s)\gamma_p\bigl(p^{-s}\bigr)^{-1},$$

where

$$\gamma_p(X)=\begin{cases}\prod_{j=0}^{n-1}(1-p^{2j}X) & \text{if } p\nmid D_H,\\ \prod_{j=0}^{[(n-1)/2]}(1-p^{4j}X) & \text{if } p\mid D_H.\end{cases}$$

Let $\widetilde{F}_{p,h}$ be a Laurent polynomial defined by

$$\widetilde{F}_{p,h}(X)=X^{-\operatorname{ord}_p D_h}F_{p,h}\bigl(p^{-2n+1}X^2\bigr).$$

Then the following functional equation holds

$$\widetilde{F}_{p,h}(X)=\widetilde{F}_{p,h}\bigl(X^{-1}\bigr)$$

(see Proposition 2.1).

Let k be an even integer and d a factor of D_H. Let

$$f(\tau)=\sum_{N=1}^{\infty}c_f(N)q^N\in S_k\bigl(\Gamma_0(d)\bigr),\qquad q=\mathbf{e}(\tau)=e^{2\pi\sqrt{-1}\tau}$$

be a primitive form with Hecke L-function:

$$\begin{aligned}L(f,s)&=\sum_{N=1}^{\infty}c_f(N)N^{-s}\\&=\prod_{p\mid d}\bigl(1-\alpha_p p^{(k-1)/2-s}\bigr)^{-1}\prod_{p\nmid d}\bigl(1-\alpha_p p^{(k-1)/2-s}\bigr)^{-1}\bigl(1-\alpha_p^{-1}p^{(k-1)/2-s}\bigr)^{-1}.\end{aligned}$$

Let us define a function $\mathrm{Lift}_n(f):\mathfrak{H}_n\to\mathbb{C}$ via

$$\mathrm{Lift}_n(f)(Z)=\sum_{h\in T_n^+}D_h^{(k-1)/2}\prod_p\widetilde{F}_{p,h}(\alpha_p)\mathbf{e}\bigl(\lambda(hZ)\bigr).$$

This is absolutely and uniformly convergent on any compact subset of $\mathfrak{H}_n$.

Our main result is the following:

Theorem 1. *Assume that $d=1$, i.e., f is a normalized Hecke eigenform in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then $\mathrm{Lift}_n(f)$ is an element of $S_{k+2n-2}(\Gamma_n)$. Moreover, $\mathrm{Lift}_n(f)$ is a common Hecke eigenform of all Hecke operators whose standard L-function is given by*

$$L^{\mathfrak{S}}\bigl(s,\mathrm{Lift}_n(f),\mathrm{st}\bigr)=\prod_{j=1}^{2n}L^{\mathfrak{S}}\left(f,s+\frac{k}{2}+n-j\right),$$

where $\mathfrak{S}$ stands for the set of all places of $\mathbb{Q}$ at which H is ramified.

The proof of Theorem 1 follows essentially the same line as Ikeda's original proofs [14,15], but a large portion of the proof is involved with the technicalities due to the lack of the knowledge of suitable generators of the group Γ_n.

We write $S_k^{\mathrm{new}}(d)$ for the space of newforms for $S_k(\Gamma_0(d))$. We define a subspace $G_\kappa(\Gamma_n)$ of $S_\kappa(\Gamma_n)$ as follows. Two matrices h and $h' \in S_n(H)$ are said to be in the same genus if there is $\beta_p \in \mathrm{GL}_n(R_p)$ satisfying $h = \beta_p^* h' \beta_p$ for every prime number p and $h = \beta_\infty^* h' \beta_\infty$ for some $\beta_\infty \in \mathrm{GL}_n(\mathbb{H})$. A function $F \in S_\kappa(\Gamma_n)$ belongs to $G_\kappa(\Gamma_n)$ if $A_F(h)$ depends only on the genus of h (cf. Corollary 8.7, Conjecture 10.3). The space $G_\kappa(\Gamma_n)$ is Hecke invariant and has a $\mathbb{C}$-basis which consists of common eigenforms for all Hecke operators. When $n = 2$, we shall show the following:

Theorem 2. *Notation being as above, we assume that $n = 2$. Then $\mathrm{Lift}_2(f)$ is a cuspidal Hecke eigenform in $G_{k+2}(\Gamma_2)$ with standard L-function*:

$$L^{\mathfrak{S}}\big(s, \mathrm{Lift}_2(f), \mathrm{st}\big) = \prod_{j=1}^{4} L^{\mathfrak{S}}\left(f, s + \frac{k}{2} + 2 - j\right).$$

Moreover, the lifting $f \mapsto \mathrm{Lift}_2(f)$ gives a bijection (up to a scalar) between Hecke eigenforms in $\bigoplus_{d \geqslant 1, d | D_H} S_k^{\mathrm{new}}(d)$ and those in $G_{k+2}(\Gamma_2)$.

In the two theorems above we include Hecke operators at prime factors of D_H though Euler factors for such prime numbers are omitted. The precise statements of the two theorems are Theorems 4.2 and 4.3, in which we determine the local components of the automorphic representation associated to $\mathrm{Lift}_n(f)$ at all places, including $\mathfrak{S}$.

The proof of Theorem 2 is entirely different from that of Theorem 1. Though the method adapted here does not apply to the case $n > 2$, we expect that when n is even, $\mathrm{Lift}_n(f)$ is a Hecke eigenform in $G_{k+2n-2}(\Gamma_n)$ for every d. If n is odd and if $d > 1$, then this is not the case, but a certain lifting is expected to exist in this case as well. Conjecture 10.1 predicts the existence of such liftings more generally for Hilbert cusp forms.

This paper is organized as follows. Section 1 sets up the notion of modular forms on quaternionic unitary groups. Section 2 is devoted to a local study of the Siegel series. In Section 3 we study the Eisenstein series on the quaternion half-space. We give an explicit expression for the functional equation of the Eisenstein series and calculate the Fourier coefficients of the Eisenstein series. As a by-product of these, we obtain the functional equation of the Siegel series. In Section 4 we state our main results. After some preparation in Section 5, we prove Theorem 1 in Section 6.

The proof of Theorem 2 relies heavily on the classical theory of the lifting from Jacobi forms to modular forms of degree two and a relation between Jacobi forms and elliptic modular forms. Under the assumptions (I) and (II) above, these have already been obtained by Krieg [23,24]. Section 7 extends Krieg's work to the setting of general quaternion algebras. To complete our picture, we connect $\mathrm{Lift}_2(f)$, through the spin group, to a certain cusp form on the orthogonal group of signature $(2, 6)$, which has already been studied by Gritsenko [10] and Sugano [34]. Section 8 elaborates on this step. We include a complete discussion of the relation of spin groups to orthogonal groups and in turn to quaternionic unitary groups since we have not been able to find a reference that contains all of the facts we need. In Section 9 we linearize our lifting, following Kohnen [20]. We discuss a description of the image of this linear map, which eventually completes the proof of Theorem 2.

We shall discuss several open problems in Section 10. In Appendix A, we investigate the automorphic L-function attached to $\mathrm{Lift}_n(f)$, using the doubling method. Finally, in Appendix B, we explain how our lifting fits into the framework of Arthur's conjecture. In particular, we describe the Arthur parameter associated to $\mathrm{Lift}_n(f)$ and discuss the conjectural structure of Arthur packets.

Notation. Let us fix the general notation that we will use. We write $\mathbb{N}$ for the set of positive integers. Throughout this paper we make the convention that the letters k and κ always denote even integers.

Let H be a quaternion algebra central over a field F, but we will mostly take $F=\mathbb{Q}$ and assume that H is definite, i.e., $H\otimes_{\mathbb{Q}}\mathbb{R}\simeq\mathbb{H}$ is the Hamilton quaternion division algebra. Let $^{\iota}$ be the main involution of H and $\nu,\tau:\mathrm{M}_n(H)\to F$ the reduced norm and the reduced trace on $\mathrm{M}_n(H)$ respectively. Put $\lambda=\tau/2$. Fix once and for all a maximal order R of H. If $F=\mathbb{Q}$, then we write D_H for the product of rational primes p for which $H_p=H\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is a division algebra.

For $x\in\mathbb{R}$, we denote by $[x]$ the Gauss bracket of x. We denote by the formal symbol ∞ the infinite place and do not use p or q for the infinite place. Set $\mathbf{e}(\tau)=\mathbf{e}_\infty(\tau)=\exp(2\pi\sqrt{-1}\tau)$ and $q^N=\mathbf{e}(N\tau)$ for $\tau\in\mathbb{C}$. Put

$$\mathbf{e}_p(x)=\exp\bigl(-2\pi\sqrt{-1}(\text{fractional part of }x)\bigr)$$

for $x\in\mathbb{Q}_p$. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_{\mathrm{f}}$ the finite part of the adele ring. Put $\mathbf{e}_{\mathbb{A}}(x)=\prod_v\mathbf{e}_v(x_v)$ for $x=(x_v)\in\mathbb{A}$. For an algebraic group G over $\mathbb{Q}$, its group of $\mathbb{Q}$-rational points, its group of $\mathbb{Q}_v$-rational points, its adele group and the finite part of the adele group are denoted by $G(\mathbb{Q})$, $G(\mathbb{Q}_v)$, $G(\mathbb{A})$ and $G(\mathbb{A}_{\mathrm{f}})$ respectively. For an adele point $a\in G(\mathbb{A})$, we denote by a_∞ (resp. $a_{\mathbf{f}}$) its infinite (resp. finite) part.

For a ring $\mathcal{O}$, let $\mathcal{O}^\times$ be the group of all its invertible elements and put $\mathrm{GL}_n(\mathcal{O})=\mathrm{M}_n(\mathcal{O})^\times$. Let $\mathbf{1}_n$ be the identity matrix of degree n and $\mathbf{0}_n$ the zero matrix of degree n. Given square matrices $a_1,\dots,a_m$, we denote by $\mathrm{diag}[a_1,\dots,a_m]$ the matrix whose diagonal blocks are equal to $a_1,\dots,a_m$ and all entries on the off-diagonal blocks are equal to zero. Assume that $\mathcal{O}$ has an involution $a\mapsto a^\iota$. For a matrix x over $\mathcal{O}$, let ${}^tx$ be the transpose of x and $x^*={}^tx^\iota$ the conjugate transpose of x. Set $z[y]=y^*zy$ for matrices y and z if it is well defined. Put $S_n(\mathcal{O})=\{x\in\mathrm{M}_n(\mathcal{O})\mid x^*=x\}$. Let $\mathcal{O}'$ be a subring of $\mathcal{O}$. We call two matrices $h,h'\in S_n(\mathcal{O})$ equivalent over $\mathcal{O}'$ with each other if there is an element $\beta\in\mathrm{GL}_n(\mathcal{O}')$ such that $h'=h[\beta]$. Define an embedding $\iota_0:\mathrm{M}_{2m}(\mathcal{O})\times\mathrm{M}_{2r}(\mathcal{O})\hookrightarrow\mathrm{M}_{2(m+r)}(\mathcal{O})$ by

$$\iota_0(g_1,g_2)\mapsto\left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2\end{array}\right),$$

writing a typical element $g_1\in\mathrm{M}_{2m}(\mathcal{O})$ in the form $\left(\begin{smallmatrix}a_1&b_1\\c_1&d_1\end{smallmatrix}\right)$ with a matrix d_1 of size m and similarly for $g_2\in\mathrm{M}_{2r}(\mathcal{O})$.

1. Modular forms on quaternionic unitary groups

Let H be a quaternion algebra over a field F and G_n a connected algebraic group defined over F whose group of D-valued points is given by

$$\left\{\alpha\in\mathrm{SL}_{2n}(H\otimes_F D)\;\middle|\;\alpha\begin{pmatrix}0&-\mathbf{1}_n\\ \mathbf{1}_n&0\end{pmatrix}\alpha^*=\begin{pmatrix}0&-\mathbf{1}_n\\ \mathbf{1}_n&0\end{pmatrix}\right\}$$

for any F-algebra D. Here we set

$$\mathrm{SL}_m(H\otimes_F D)=\bigl\{\alpha\in\mathrm{GL}_m(H\otimes_F D)\;\big|\;\nu(\alpha)=1\bigr\},$$

where ν denotes the reduced norm on $\mathrm{M}_m(H\otimes_F D)$. It is important that there is an accidental isomorphism

$$G_1(F)\simeq\bigl\{(\alpha,\gamma)\in H^\times\times\mathrm{GL}_2(F)\;\big|\;\nu(\alpha)\det\gamma=1\bigr\}/\bigl\{\bigl(z,z^{-1}\bigr)\;\big|\;z\in F^\times\bigr\}.\tag{1.1}$$

Lemma 1.1. *There exists a polynomial map $P: S_n(H) \to F$ defined over F such that $\nu(X) = P(X)^2$ for every $X \in S_n(H)$.*

Proof. Representing X by

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2^* & x_3 \end{pmatrix} \quad \bigl(x_1 \in F,\ x_2 \in H^{n-1},\ x_3 \in S_{n-1}(H)\bigr),$$

we have

$$\begin{pmatrix} 1 & -x_1^{-1}x_2 \\ 0 & \mathbf{1}_{n-1} \end{pmatrix}^* X \begin{pmatrix} 1 & -x_1^{-1}x_2 \\ 0 & \mathbf{1}_{n-1} \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_3 - x_1^{-1}x_2^*x_2 \end{pmatrix}.$$

It follows that

$$\nu(X) = x_1^2 \nu\bigl(x_3 - x_1^{-1}x_2^*x_2\bigr).$$

Since entries of the matrix $x_3 - x_1^{-1}x_2^*x_2 \in S_{n-1}(H)$ are rational functions in the coordinates of X with respect to any given base of $S_n(H)$ over F, the induction argument yields a rational function $Q(X)$ with coefficients in F such that $\nu(x_3 - x_1^{-1}x_2^*x_2) = Q(X)^2$. Since $(x_1 Q(X))^2$ is a polynomial map, so is $x_1 Q(X)$. Hence $P(X) = x_1 Q(X)$ works. $\square$

We write Paf for the unique polynomial map that satisfies the condition of Lemma 1.1 and such that $\mathrm{Paf}(\mathbf{1}_n) = 1$. Record a simple identity

$$\mathrm{Paf}\, h[\beta] = \nu(\beta)\, \mathrm{Paf}\, h \quad \bigl(h \in S_n(H),\ \beta \in \mathrm{GL}_n(H)\bigr).$$

We hereafter take $F = \mathbb{Q}$ and assume that H is definite. Fix a maximal order R of H. The set of semi-integral hermitian matrices is defined by

$$T_n = \bigl\{h \in S_n(H) \bigm| \lambda(h\beta) \in \mathbb{Z} \text{ for every } \beta \in S_n(R)\bigr\}.$$

We call an element $h \in S_n(\mathbb{H})$ positive definite if $h[x] > 0$ for all $x \in \mathbb{H}^n \setminus \{0\}$. Let $\mathcal{P}_n$ be the set of positive definite hermitian matrices of size n over $\mathbb{H}$. Put

$$S_n^+(H) = S_n(H) \cap \mathcal{P}_n, \qquad T_n^+ = T_n \cap \mathcal{P}_n.$$

We define the quaternion half-space of degree n by

$$\mathfrak{H}_n = \bigl\{Z = X + \sqrt{-1}Y \in S_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \bigm| X \in S_n(\mathbb{H}),\ Y \in \mathcal{P}_n\bigr\}.$$

The group $G_n(\mathbb{R})$ acts transitively on $\mathfrak{H}_n$ by $\alpha Z = (aZ + b)(cZ + d)^{-1}$ for $Z \in \mathfrak{H}_n$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_n(\mathbb{R})$. We set

$$j(\alpha, Z) = \nu(cZ + d), \qquad j_\kappa(\alpha, Z) = j(\alpha, Z)^{\kappa/2}.$$

Recall that κ is tacitly assumed to be even. Define the modular group by setting

$$\Gamma_n = \mathrm{GL}_{2n}(R) \cap G_n(\mathbb{Q}).$$

Put $\mathbf{i} = \sqrt{-1}\mathbf{1}_n$. Let R_p be the closure of R in $H_p = H \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We define a maximal compact subgroup $C_{n,v}$ of $G_n(\mathbb{Q}_v)$ by

$$C_{n,\infty} = \{\alpha \in G_n(\mathbb{R}) \mid \alpha(\mathbf{i}) = \mathbf{i}\} = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{H}) \;\middle|\; a + \sqrt{-1}b \in U(2n)\right\},$$
$$C_{n,p} = G_n(\mathbb{Q}_p) \cap \mathrm{GL}_{2n}(R_p).$$

We here put $U(2n) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid g \cdot {}^t\overline{g} = \mathbf{1}_{2n}\}$ and utilize a suitable isomorphism $\mathrm{M}_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{M}_{2n}(\mathbb{C})$. Set $C_n = \prod_v C_{n,v}$.

Define algebraic groups Q_n and U_n by

$$Q_n(\mathbb{Q}) = \{m(a) = \mathrm{diag}[a, (a^{-1})^*] \mid a \in \mathrm{GL}_n(H)\},$$
$$U_n(\mathbb{Q}) = \left\{n(b) = \begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix} \;\middle|\; b \in S_n(H)\right\}.$$

Let $P_n = Q_n U_n$. Recall the Iwasawa decomposition

$$G_n(\mathbb{A}) = P_n(\mathbb{A}) C_n$$

(see [6, Theorem 3.3]). It is important to note that

$$\nu(\mathrm{GL}_n(\mathbb{H})) = \mathbb{R}_+^\times, \qquad \nu(\mathrm{GL}_n(H_p)) = \mathbb{Q}_p^\times, \qquad \nu(\mathrm{GL}_n(H)) = \mathbb{Q}_+^\times$$

(see [35]). Provided that $n \geqslant 2$, the strong approximation property of the adele group of $\mathrm{SL}_n(H)$ combined with these yields

$$Q_n(\mathbb{A}) = Q_n(\mathbb{Q}) Q_n(\mathbb{R}) \prod_p m(\mathrm{GL}_n(R_p)).$$

If $n = 1$, then this is not the case in general. This combined with the Iwasawa decomposition above gives

$$G_n(\mathbb{A}) = P_n(\mathbb{Q}) G_n(\mathbb{R}) C_n \tag{1.2}$$

for $n \geqslant 2$. It follows immediately that

$$G_n(\mathbb{Q}) = P_n(\mathbb{Q}) \Gamma_n. \tag{1.3}$$

The isomorphism (1.1) guarantees that (1.3) remains valid when $n = 1$. Moreover, it follows easily that

$$\Gamma_1 = \{\varepsilon \mid \varepsilon \in R^\times\} \cdot \mathrm{SL}_2(\mathbb{Z}). \tag{1.4}$$

For $\alpha \in G_n(\mathbb{R})$ and a $\mathbb{C}$-valued function F on $\mathfrak{H}_n$, we define $F|_\kappa \alpha : \mathfrak{H}_n \to \mathbb{C}$ by

$$F|_\kappa \alpha(Z) = j_\kappa(\alpha, Z)^{-1} F(\alpha Z).$$

When $n = 1$, we put

$$F|_k \alpha(\tau) = (\det \alpha)^{k/2} (c\tau + d)^{-k} F((a\tau + b)(c\tau + d)^{-1})$$

for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g > 0\}$.

Definition 1.1. When $n \geqslant 2$, a holomorphic function F on $\mathfrak{H}_n$ is called a modular form of weight κ if $F|_\kappa \gamma = F$ for every $\gamma \in \Gamma_n$. When $n = 1$, we impose the usual holomorphy condition at the cusp. A modular form F is called a cusp form if $F|_\kappa \beta$ has a Fourier expansion of the form

$$F|_\kappa \beta(Z) = \sum_{h \in S_n^+(H)} C(h)\mathbf{e}\bigl(\lambda(hZ)\bigr)$$

for all $\beta \in G_n(\mathbb{Q})$. The space of modular (resp. cusp) forms of weight κ is denoted by $M_\kappa(\Gamma_n)$ (resp. $S_\kappa(\Gamma_n)$).

The group $G_1(\mathbb{R})$ acts on $\mathfrak{H}_1$ through $\mathrm{GL}_2^+(\mathbb{R})$ by (1.1), and as such, we get $M_k(\Gamma_1) = M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\Gamma_1) = S_k(\mathrm{SL}_2(\mathbb{Z}))$ by (1.4).

We will sometimes work adelically and restate the definition of modular forms in terms of adele groups.

Definition 1.2. The symbol $\mathfrak{M}_\kappa^n$ (resp. $\mathfrak{S}_\kappa^n$) stands for the space consisting of all continuous functions $\mathcal{F}$ on $G_n(\mathbb{A})$ satisfying the following conditions:

- for $\gamma \in G_n(\mathbb{Q})$, $x \in G_n(\mathbb{A})$, $w \in C_n$, we have

$$\mathcal{F}(\gamma x w) = \mathcal{F}(x)(\det \mathbf{w}_\infty)^{\kappa/2},$$

  where w_∞ corresponds to $\mathbf{w}_\infty \in U(2n)$ via the map $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mapsto a + b\sqrt{-1}$;
- for each $\delta \in G_n(\mathbb{A}_\mathrm{f})$, we define a function $\mathcal{F}_\delta : \mathfrak{H}_n \to \mathbb{C}$ by

$$\mathcal{F}_\delta|_\kappa g(\mathbf{i}) = \mathcal{F}(g\delta), \quad g \in G_n(\mathbb{R}).$$

  Then $\mathcal{F}_\delta$ is holomorphic and possesses a Fourier expansion of the form

$$\mathcal{F}_\delta(Z) = \sum_{h \in S_n(H)} C(h)\mathbf{e}\bigl(\lambda(hZ)\bigr),$$

  where $C(h) = 0$ unless h is positive semi-definite (resp. positive definite).

Here, the function $\mathcal{F}_\delta$ is well defined since $(\det \mathbf{w}_\infty)^{\kappa/2} = j_\kappa(w_\infty, \mathbf{i})^{-1}$.

Remark 1.3.

(1) It is noteworthy that we can restrict β to $\mathbf{1}_{2n}$ in Definition 1.1 by virtue of (1.3).
(2) Since $\mathcal{F}$ is determined by its restriction to $G_n(\mathbb{R})$ when $n \geqslant 2$ on account of (1.2), the space $\mathfrak{M}_\kappa^n$ (resp. $\mathfrak{S}_\kappa^n$) can be identified with $M_\kappa(\Gamma_n)$ (resp. $S_\kappa(\Gamma_n)$) through $\mathcal{F} \to \mathcal{F}_{\mathbf{1}_{2n}}$.

2. Siegel series for quaternion hermitian forms

Fix a prime number p. Put $H_p = H \otimes_\mathbb{Q} \mathbb{Q}_p$ and $R_p = R \otimes_\mathbb{Z} \mathbb{Z}_p$. Let us set

$$T_{n,p} = \bigl\{ h \in S_n(H_p) \bigm| \lambda(h\beta) \in \mathbb{Z}_p \text{ for every } \beta \in S_n(R_p) \bigr\},$$

where λ is the half of the reduced trace on $M_n(H_p)$. For any element $h \in T_{n,p}$ we put $D_h = D_H^{[n/2]} \operatorname{Paf} h$. Observe that $D_h \in \mathbb{Z}_p$. We call h regular if $\operatorname{ord}_p D_h = 0$.

The Siegel series associated to h is defined by

$$b_p(h,s)=\sum_{\beta\in S_n(H_p)/S_n(R_p)}\mathbf{e}_p\bigl(-\lambda(h\beta)\bigr)\nu[\beta]^{-s/2},$$

where the quantity $\nu[\beta]$ is defined in the following way. Let $|\cdot|_p$ denote the normalized valuation of $\mathbb{Q}_p$. Taking $c\in \mathrm{M}_n(R_p)\cap \mathrm{GL}_n(H_p)$ and $d\in \mathrm{M}_n(R_p)$ so that $\beta=c^{-1}d$ and $(c\ d)$ is primitive, we put $\nu[\beta]=|\nu(c)|_p^{-1}$. Recall that we call a matrix $x\in \mathrm{M}_{mn}(R_p)$ $(m\leqslant n)$ primitive if there exists an element $\beta\in \mathrm{M}_n(R_p)$ such that $x\beta=(\mathbf{1}_m\ 0)$. Then $\nu[\beta]$ is well defined and is equal to $[\beta R_p^n+R_p^n:R_p^n]^{1/2}$ (see [6, Corollary 2.3, Lemma 2.4] or [31, §3]). Note that $\mathrm{ord}_p\,\nu[\beta]$ is a power of p^2 if $\beta\in S_n(H_p)$.

The series $b_p(h,s)$ defines a formal Dirichlet series in the variable s, which converges absolutely for sufficiently large $\Re s$ and becomes a rational function of p^{-s}. If h is nondegenerate, then the sum stabilizes and consequently, $b_p(h,s)$ is a polynomial of p^{-s}. More precisely, if we put $e=\mathrm{ord}_p\,D_h$, then

$$b_p(h,s)=\sum_{\beta\in(p^{-2e-1}S_n(R_p))/S_n(R_p)}\mathbf{e}_p\bigl(-\lambda(h\beta)\bigr)\nu[\beta]^{-s/2}. \tag{2.1}$$

Assume that p and D_H are coprime. Fix an isomorphism $f_p:H_p\simeq \mathrm{M}_2(\mathbb{Q}_p)$ such that $f_p(R_p)=\mathrm{M}_2(\mathbb{Z}_p)$. Define an isomorphism $g_p:\mathrm{M}_n(H_p)\to \mathrm{M}_{2n}(\mathbb{Q}_p)$ by $g_p((x_{ij}))=(f_p(x_{ij}))$. Since $f_p(x^\iota+x)=\tau(x)\cdot\mathbf{1}_2=\mathrm{tr}(f_p(x))\cdot\mathbf{1}_2$, we have

$$f_p\bigl(x^\iota\bigr)=J^{-1t}f_p(x)J,\qquad J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Put $B_n=\mathrm{diag}[J,\dots,J]\in \mathrm{M}_{2n}(\mathbb{Q}_p)$. Then $g_p(X^*)=B_n{}^tg_p(X)B_n^{-1}$, where ${}^tg_p(X)$ is the transpose of $g_p(X)$ as a matrix of size $2n$. Therefore,

$$g_p\bigl(S_n(H_p)\bigr)=B_nS'_{2n},\qquad g_p\bigl(S_n(R_p)\bigr)=B_nS'_{2n}\cap \mathrm{M}_{2n}(\mathbb{Z}_p)$$

where S'_{2n} denotes the space of skew symmetric matrices over $\mathbb{Q}_p$ of degree $2n$. In this case $b_p(h,s)$ belongs to the class of local Dirichlet series investigated in [30], and the proof of (2.1) is given in [30, Proposition 14.3, Lemma 14.8]. More importantly, the map f_p induces an isomorphism of $G_n(\mathbb{Q}_p)$ onto

$$\mathrm{SO}(2n,2n;\mathbb{Q}_p)=\left\{x\in \mathrm{SL}_{4n}(\mathbb{Q}_p)\ \middle|\ x\begin{pmatrix}0&\mathbf{1}_{2n}\\\mathbf{1}_{2n}&0\end{pmatrix}{}^tx=\begin{pmatrix}0&\mathbf{1}_{2n}\\\mathbf{1}_{2n}&0\end{pmatrix}\right\},$$

whence G_n is an inner form of $\mathrm{SO}(2n,2n)$.

We omit the proof of (2.1) for each prime factor p of D_H as it can be proven similarly. A different kind of proof can be found in [7].

We define the polynomial γ_p by

$$\gamma_p(X)=\begin{cases}\prod_{j=0}^{n-1}(1-p^{2j}X) & \text{if } p\nmid D_H,\\[2ex] \prod_{j=0}^{[(n-1)/2]}(1-p^{4j}X) & \text{if } p|D_H.\end{cases}$$

We shall prove the following proposition in Section 3.

Proposition 2.1. *Suppose that $h \in S_n(H_p)$ is nondegenerate. Then $b_p(h,s)$ is a polynomial in p^{-s} with coefficients in $\mathbb{Z}$ and constant term 1, which is divisible by $\gamma_p(p^{-s})$. Put*

$$F_{p,h}(p^{-s}) = b_p(h,s)\gamma_p(p^{-s})^{-1}, \qquad \widetilde{F}_{p,h}(X) = X^{-\operatorname{ord}_p D_h} F_{p,h}(p^{-2n+1}X^2).$$

Then $\widetilde{F}_{p,h}$ satisfies the following functional equation

$$\widetilde{F}_{p,h}(X) = \widetilde{F}_{p,h}(X^{-1}). \tag{2.2}$$

It is well known that two matrices of $S_n(H_p)$ are equivalent over H_p if and only if they have the same rank. We write $h \approx_p h'$ if h and h' are equivalent over R_p (see Notation). When p and D_H are coprime, we identify $S_n(H_p)$ with S'_{2n} through g_p. When D_H is divisible by p, we write $\mathfrak{P}_p$ for the maximal ideal of R_p and fix a generator ϖ_p of $\mathfrak{P}_p$. Invariants for equivalence over R_p are given as follows:

Lemma 2.2. *Let $\sigma \in S_n(H_p) \cap \mathrm{GL}_n(H_p)$.*

(1) *Assume that p and D_H are coprime. Then $\sigma \approx_p \mathrm{diag}[\sigma_1, \dots, \sigma_n]$, where each σ_i is of the form $\sigma_i = p^{a_i} J$ for some $a_i \in \mathbb{Z}$. Moreover, the equivalent class of σ over R_p is uniquely determined by the set $\{\sigma_1, \dots, \sigma_n\}$.*
(2) *Assume that p divides D_H. Then $\sigma \approx_p \mathrm{diag}[\sigma_1, \dots, \sigma_r]$ with σ_i of size 1 or 2, where each σ_i of size 1 (resp. 2) is of the form p^{b_i} (resp. $\sigma_p(c_i) = \begin{pmatrix} 0 & \varpi_p^{c_i} \\ (\varpi_p^{\iota})^{c_i} & 0 \end{pmatrix}$) for some $b_i \in \mathbb{Z}$ (resp. odd $c_i \in \mathbb{Z}$). Moreover, the equivalent class of σ over R_p is uniquely determined by the set $\{\sigma_1, \dots, \sigma_r\}$.*

Proof. The first assertion is nothing but Lemma 13.3(2) of [30]. The second assertion amounts to [16, Propositions 4.3, 6.1, Theorem 6.2]. □

For any nonzero element $\sigma \in T_{n,p}$ we put $\epsilon_p(\sigma) = \max\{a \in \mathbb{Z} \mid p^{-a}\sigma \in T_{n,p}\}$. We will use the following corollary at the end of Section 8.

Corollary 2.3. *Let $h \in T_{2,p} \cap \mathrm{GL}_2(H_p)$. Then the equivalent class of h over R_p is uniquely determined by $\operatorname{ord}_p D_h$ and $\epsilon_p(h)$.*

Proof. Put $a = \operatorname{ord}_p D_h$ and $b = \epsilon_p(h)$. Observe that $a \geqslant 2b$. We can easily see by Lemma 2.2 that $h \approx_p \mathrm{diag}[p^{a-b}J, p^b J]$, provided that p and D_H are coprime. Suppose that p divides D_H. Then $h \approx_p \sigma_p(2b-1)$ or $h \approx_p \mathrm{diag}[p^{a-b-1}, p^b]$ according as $a = 2b$ or $a \geqslant 2b+1$. □

We return to the global situation and conclude this section by stating the following result, which will be needed later.

Lemma 2.4. *Let $p_1, \dots, p_r$ be rational primes. For each i, fix an element $h_i \in T_{n,p_i} \cap \mathrm{GL}_n(H_{p_i})$. Then there is $h \in T_n^+$ satisfying the following conditions*:

(i) *h is regular at all rational primes different from $p_1, \dots, p_r$;*
(ii) *for each i, there exists $\alpha_i \in \mathrm{GL}_n(R_{p_i})$ such that $h = h_i[\alpha_i]$.*

In particular, for each $N \in \mathbb{N}$, we can find an element $h \in T_n^+$ with $D_h = N$.

Proof. By virtue of Lemma 2.2, it suffices to consider the case $n = 1$ or 2. Clearly, we have only to treat the case $n = 2$. Note that

$$T_2 = \left\{ \begin{pmatrix} a & b \\ b^\iota & c \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z},\ b \in \widetilde{R} \right\},$$

where

$$\widetilde{R} = \left\{ x \in H \,\middle|\, \tau(x\beta) \in \mathbb{Z} \text{ for every } \beta \in R \right\}.$$

According as p divides D_H or not, the set $\widetilde{R}_p = \widetilde{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ equals $\mathfrak{P}_p^{-1}$ or R_p, and regular elements of $T_{2,p}$ are equivalent to $\sigma_p(-1)$ or $\mathrm{diag}[J, J]$ over R_p.

Note that $D_{h_i[\alpha]} = \nu(\alpha) D_{h_i}$ and $\nu(\mathrm{GL}_2(R_p)) = \mathbb{Z}_p^\times$. Enlarging the set $\{h_1, \dots, h_r\}$ by adding regular elements and replacing h_i by a suitable $\mathrm{GL}_2(R_p)$-conjugate of itself if necessary, we may assume the following conditions:

(a) the set $\{p_1, \dots, p_r\}$ contains all prime factors of D_H;
(b) $D_{h_1} = \cdots = D_{h_r} (=: D) \in \mathbb{N}$;
(c) $\mathrm{ord}_p D = 0$ for all rational primes p distinct from $p_1, \dots, p_r$.

Put $h' = \mathrm{diag}[D_H^{-1} D, 1]$. For each i, choose an element $X_{p_i} \in \mathrm{GL}_2(H_{p_i})$ so that $h' = h_i[X_{p_i}]$. Since $\mathrm{Paf}\, h' = D_H^{-1} D = \mathrm{Paf}\, h_i$ by (b), we have $X_{p_i} \in \mathrm{SL}_2(H_{p_i})$. The strongly approximation property of the adele group of $\mathrm{SL}_2(H)$ gives an element $X \in \mathrm{SL}_2(H)$ which belongs to $\mathrm{SL}_2(R_p)$ for primes p other than $p_1, \dots, p_r$ and such that $X_{p_i} X^{-1} \in \mathrm{SL}_2(R_{p_i})$ for every i. Put $h = h'[X^{-1}]$. Then h is positive definite. Since h' satisfies (i) by (a) and (c), so does h. We have $h = h_i[X_{p_i} X^{-1}]$ for every i, whence h satisfies (ii). The remaining part is obvious since $\{D_h \mid h \in T_{2,p}\} = \mathbb{Z}_p$ for each prime p. $\square$

3. Fourier coefficients of Eisenstein series

Let χ be a unitary character of $\mathbb{A}^\times/\mathbb{Q}^\times$. For $s \in \mathbb{C}$ let $I_n(s, \chi)$ be the global degenerate principal series representation of $G_n(\mathbb{A})$ on the space of all smooth right C_n-finite functions Φ on $G_n(\mathbb{A})$ satisfying

$$\Phi(pg) = \chi\big(\nu(a)\big) \big|\nu(a)\big|_{\mathbb{A}}^{s+n-1/2} \Phi(g)$$

for $p = m(a)n(b) \in P_n(\mathbb{A})$ and $g \in G_n(\mathbb{A})$, where $|\cdot|_{\mathbb{A}}$ denotes the module of $\mathbb{A}^\times$. We call a C_n-finite function $\Phi = \Phi^{(s)}$ on $\mathbb{C} \times I_n(s, \chi)$ a holomorphic section if Φ is holomorphic with respect to s and $\Phi^{(s)} \in I_n(s, \chi)$ for each $s \in \mathbb{C}$. Provided that Φ is a holomorphic section, we define an Eisenstein series $E_\Phi(g)$ by

$$E_\Phi(g) = \sum_{\gamma \in P_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} \Phi(\gamma g).$$

Such series converges absolutely for $\Re s > n - \frac{1}{2}$. The general results of Langlands state that it can be continued to a meromorphic function in s on the whole plane satisfying a functional equation (see [1]).

Note that $I_n(s, \chi)$ is a restricted tensor product $\bigotimes_v I_{n,v}(s, \chi_v)$ of the corresponding local induced representations. When χ_v is trivial, we write $I_{n,v}(s, \chi_v)$ simply as $I_{n,v}(s)$. Writing any $g \in G_n(\mathbb{A})$ as $g = m(a)n(b)w \in Q_n(\mathbb{A}) U_n(\mathbb{A}) C_n$, we set

$$\varepsilon_{\kappa,s}(g) = \big|\nu(a)\big|_{\mathbb{A}}^{(\kappa+s)/2} (\det \mathbf{w}_\infty)^{\kappa/2}$$

for $\kappa \in 2\mathbb{Z}$. This is an element of $I_n(s_0)$, where $s_0 = (\kappa + s + 1)/2 - n$. The section $\varepsilon_{\kappa,s}$ has the form $\bigotimes_v \varepsilon_{\kappa,s,v}$, where local sections $\varepsilon_{\kappa,s,v}$ of $I_{n,v}(s_0)$ are determined by the following conditions:

- the restriction of $\varepsilon_{\kappa,s,\infty}$ to $C_{n,\infty} \simeq U(2n)$ is a character $\mathbf{w} \mapsto (\det \mathbf{w})^{\kappa/2}$;
- $\varepsilon_{\kappa,s,p} = \varepsilon_{\kappa+s,p}$ is the normalized $C_{n,p}$-invariant section.

We consider the Eisenstein series of the following type:

$$\widetilde{E}^n_\kappa(g,s) = E_{\varepsilon_{\kappa,s}}(g) = \sum_{\gamma \in P_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} \varepsilon_{\kappa,s}(\gamma g).$$

It might be useful to recall the relation between the adelic Eisenstein series on $G_n(\mathbb{A})$ and the classical Eisenstein series on $\mathfrak{H}_n$ defined as follows. For $\alpha \in G_n(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, we set

$$J_{\kappa,s}(\alpha, Z) = j_\kappa(\alpha, Z)\big|j(\alpha, Z)\big|^{s/2}.$$

The Eisenstein series $E^n_\kappa(Z,s)$ on $\mathfrak{H}_n$ is defined by

$$E^n_\kappa(Z,s) = \sum_{\gamma \in P_n(\mathbb{Q}) \cap \Gamma_n \backslash \Gamma_n} J_{\kappa,s}(\gamma, Z)^{-1}.$$

Write $\alpha = m(a)n(b)w \in Q_n(\mathbb{R})U_n(\mathbb{R})C_{n,\infty}$ and set $Z = \alpha(\mathbf{i})$. Then we have

$$\nu(a) = \operatorname{Paf} \Im Z, \qquad j(\alpha, \mathbf{i}) = \nu(a)^{-1}(\det \mathbf{w})^{-1},$$

where $\Im Z$ stands for the imaginary part of Z. These give rise to

$$\varepsilon_{\kappa,s,\infty}(\gamma\alpha) = \big(\operatorname{Paf} \Im(\gamma Z)\big)^{s/2} j_\kappa(\gamma\alpha, \mathbf{i})^{-1} = (\operatorname{Paf} \Im Z)^{s/2} J_{\kappa,s}(\gamma, Z)^{-1} j_\kappa(\alpha, \mathbf{i})^{-1}.$$

It follows from (1.3) that

$$\widetilde{E}^n_\kappa(\alpha,s) = (\operatorname{Paf} \Im Z)^{s/2} E^n_\kappa(Z,s) j_\kappa(\alpha, \mathbf{i})^{-1}.$$

For later use, we explicate the functional equation of $E^n_\kappa(Z,s)$ under the simplifying assumption $\kappa = 0$. Let $M(s_0) : I_n(s_0) \to I_n(-s_0)$ be the global intertwining operator, which we recall is

$$M(s_0)\varepsilon_{0,s}(g) = \int_{U_n(\mathbb{A})} \varepsilon_{0,s}\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} ug\right) du$$

for $\Re s_0 > n - \frac{1}{2}$, where du is the Haar measure on $U_n(\mathbb{A})$ normalized so that $U_n(\mathbb{A}/\mathbb{Q})$ has volume 1.

Lemma 3.1. *Let $d\mu = \prod_v d\mu_v$ be the Haar measure on $S_n(H \otimes_{\mathbb{Q}} \mathbb{A})$ obtained by taking $d\mu_\infty = dx$ and $\mu_p(S_n(R_p)) = 1$, where the Lebesgue measure dx on $S_n(\mathbb{H})$ is defined by viewing $S_n(\mathbb{H})$ as $\mathbb{R}^n \times \mathbb{H}^{n(n-1)/2} \simeq \mathbb{R}^{n(2n-1)}$ using the standard quaternion units $\{1, i, j, k\}$ of $\mathbb{H}$. Then $du = (4D_H^{-1})^{n(n-1)/2}\, d\mu$.*

Proof. Lemma 5.4 of [31] proves the skew-hermitian analog of Lemma 3.1 over arbitrary number fields. We can easily deduce our lemma from its proof. □

Since $M(s_0)\varepsilon_{0,s}$ is invariant for the right action of C_n, we have

$$M(s_0)\varepsilon_{0,s}(g) = \gamma(s)\varepsilon_{0,4n-2-s}(g),$$

where $\gamma(s)=(4D_H^{-1})^{n(n-1)/2}\prod_v \gamma_v(s)$ is given by

$$\gamma_v(s)=\int\limits_{S_n(H_v)} \varepsilon_{0,s,v}\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} n(x)\right) d\mu_v(x).$$

Note that

$$\gamma_\infty(s)=\int\limits_{S_n(\mathbb{H})} \left|\operatorname{Paf}(x+\mathbf{i})\right|^{-s} dx = 2^{n(2n-s)}\pi^{n(2n-1)}\frac{\Gamma_n(s-2n+1)}{\Gamma_n(\frac{s}{2})^2}$$

by [29, (1.31)], where $\Gamma_n(s)=\pi^{n(n-1)}\prod_{i=0}^{n-1}\Gamma(s-2i)$. It is easy to see that

$$\varepsilon_{s,p}\begin{pmatrix} \mathbf{1}_n & 0 \\ x & \mathbf{1}_n \end{pmatrix} = \nu[x]^{-s/2} \tag{3.1}$$

for $x\in S_n(H_p)$. Thus we obtain

$$\gamma_p(s)=b_p(0,s)=\begin{cases} \prod_{i=0}^{n-1}\frac{1-p^{-s+4i}}{1-p^{-s+2n-1+2i}} & \text{if } p|D_H, \\ \prod_{i=0}^{n-1}\frac{1-p^{-s+2i}}{1-p^{-s+2n-1+2i}} & \text{if } p\nmid D_H \end{cases}$$

by [30, Proposition 15.4, (15.4.2)] and [31, Proposition 3.5]. It follows that

$$\widetilde{E}_0^n(g,s)=\gamma(s)\widetilde{E}_0^n(g,4n-2-s).$$

To be more precise, put

$$\begin{aligned}\mathbf{E}_0^n(Z,s)&=D_H^{([n/2]s)/2}(\operatorname{Paf}\Im Z)^{s/2}E_0^n(Z,s)\\ &\quad\times\prod_{i=1}^{[n/2]}\left\{(s-4i+2)\prod_{p|D_H}\left(1-p^{-s+4i-2}\right)\right\}\prod_{j=0}^{n-1}\xi(s-2j),\end{aligned}$$

where $\xi(s)=\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Then one can find

$$\mathbf{E}_0^n(Z,s)=\mathbf{E}_0^n(Z,4n-2-s). \tag{3.2}$$

Next we present formulas for the Fourier coefficients of $E_\kappa^n(Z,s)$. For $Y,\ h\in\mathcal{P}_n$ we define $\Xi(Y,h;s,s')$ for sufficiently large $\Re(s+s')$, by the integral

$$\Xi\left(Y,h;s,s'\right)=\int\limits_{S_n(\mathbb{H})}\mathbf{e}\left(-\lambda(hx)\right)\operatorname{Paf}(x+\sqrt{-1}Y)^{-s}\operatorname{Paf}(x-\sqrt{-1}Y)^{-s'}dx$$

and by meromorphic continuation otherwise. For basic properties of the function $\Xi(Y,h;s,s')$, the reader is referred to [29]. We note here only that

$$\Xi(Y,h;s,0)=2^{n(1-\ell)}\left(-2\pi\sqrt{-1}\right)^{ns}\Gamma_n(s)^{-1}(\operatorname{Paf}h)^{s-\ell}\mathbf{e}\left(\sqrt{-1}\lambda(Yh)\right), \tag{3.3}$$

$$\Xi\left(Y,h;s,s'\right)=\left(\frac{\pi^n\operatorname{Paf}h}{\operatorname{Paf}Y}\right)^{s+s'-\ell}\frac{\Gamma_n(\ell-s')}{\Gamma_n(s)}\Xi\left(Y,h;\ell-s',\ell-s\right), \tag{3.4}$$

where $\ell=2n-1$. Let $Z=X+\sqrt{-1}Y\in\mathfrak{H}_n$. Now $E^n_\kappa(Z,s)$ has a Fourier expansion of the form

$$E^n_\kappa(Z,s)=\sum_{h\in T_n}c^n_\kappa(h;Y,s)\mathbf{e}\left(\lambda(hX)\right)$$

with

$$c^n_\kappa(h;Y,s)=\left(4D_H^{-1}\right)^{n(n-1)/2}\Xi\left(Y,h;\kappa+\frac{s}{2},\frac{s}{2}\right)\prod_p b_p(h,\kappa+s)$$

for $h\in T_n^+$. The proof for this formula is straightforward and is omitted.

Proof of Proposition 2.1. Our method of proof is substantially the same as that of [5]. Observe first that

$$\prod_{i=1}^{[n/2]}\left\{\prod_{p|D_H}\left(1-p^{-s+4i-2}\right)\right\}\prod_{j=0}^{n-1}\zeta(s-2j)\prod_p b_p(h,s)=\prod_p\frac{b_p(h,s)}{\gamma_p(p^{-s})}, \tag{3.5}$$

where the right-hand side is actually a finite product by Proposition 14.9 of [30]. Combining (3.2), (3.4) and the Fourier coefficient formula just described, we can derive that

$$\prod_p\frac{b_p(h,s)}{\gamma_p(p^{-s})}=D_h^{2n-1-s}\prod_p\frac{b_p(h,4n-2-s)}{\gamma_p(p^{-(4n-2-s)})}.$$

Recalling that $b_p(h,s)=\gamma_p(p^{-s})F_{p,h}(p^{-s})\in\mathbb{C}[p^{-s}]$, we see that

$$\gamma_p\left(p^{-4n+2}X^{-1}\right)F_{p,h}(X)\in\mathbb{C}\left[X,X^{-1}\right].$$

It turns out that $F_{p,h}$ is a polynomial of degree $\operatorname{ord}_p D_h$ since $\gamma_p(X)$ and $\gamma_p(p^{-4n+2}X^{-1})$ are coprime as elements of $\mathbb{C}[X,X^{-1}]$. As the constant term of $F_{p,h}$ is 1, we have $F_{p,h}=1$ whenever h is regular. Since Lemma 2.4 allows us to take $h'\in T_n^+$ which is equivalent to h over R_p and such that the localizations of h' at q are regular for all $q\neq p$, the identity (2.2) readily drops out.

Let L be a field generated over $\mathbb{Q}$ by all p-power roots of unity and $\mathfrak{R}$ the integer ring of L. Clearly $F_{p,h}\in\mathfrak{R}[X]$. Given $a\in\mathbb{Z}_p^\times$, let σ_a be an automorphism of L over $\mathbb{Q}$ determined by $\mathbf{e}(p^{-m})\mapsto\mathbf{e}(ap^{-m})$. Since $b_p(h,s)^{\sigma_a}=b_p(ah,s)=b_p(h,s)$, we conclude that $F_{p,h}\in\mathbb{Z}[X]$. $\square$

Remark 3.1.

(1) Under the assumptions (I) and (II), the functional equation (3.2) and the Fourier coefficient formula were given in [18].

(2) These results are basic when $n=1$ as E^1_k is a well-known function

$$E^1_k(\tau,s)=2^{-1}\sum_{(c,d)\in\mathbb{Z}^2,\,(c,d)=1}(c\tau+d)^{-k}|c\tau+d|^{-s}$$

in view of (1.3) and (1.4). In this case we have

$$\widetilde{F}_{p,h} = l_e, \qquad l_e(X) = \frac{X^{e+1} - X^{-e-1}}{X - X^{-1}}, \qquad e = \operatorname{ord}_p h.$$

(3) The restriction to $F = \mathbb{Q}$ is only for convenience. As was pointed out by Watanabe, we can reprove (2.2) by rewriting the functional equation of the generalized Whittaker functional given in [37] (see the remark after the proof of [17, Proposition 3.1] and [15, §3]). This method works at least over any nonarchimedean local fields of characteristic zero.

(4) The assertions of Proposition 2.1 except for (2.2) follows also from the formula for $b_p(h,s)$ given by Feit [7,8] (see (9.1)).

Assume that $s = 0$ and $\kappa > 4n - 2$. Put $E^n_\kappa(Z) = E^n_\kappa(Z, 0)$ and

$$\mathcal{E}^{(n)}_\kappa(Z) = 2^{-n} \prod_{i=1}^{[n/2]} \left\{ \prod_{p|D_H} \bigl(1 - p^{\kappa-4i+2}\bigr) \right\} \prod_{i=0}^{n-1} \zeta(1 + 2i - \kappa) E^n_\kappa(Z).$$

Proposition 3.2. *For $h \in T^+_n$, the hth Fourier coefficient of $\mathcal{E}^{(n)}_\kappa$ is given by*

$$D_h^{(\kappa+1)/2-n} \prod_p \widetilde{F}_{p,h}\bigl(p^{(\kappa+1)/2-n}\bigr).$$

Proof. From (3.3) and (3.5), the hth Fourier coefficient of $\mathcal{E}^{(n)}_\kappa$ equals

$$2^{-n}\bigl(4D_H^{-1}\bigr)^{n(n-1)/2} 2^{2n(1-n)} (-2\pi\sqrt{-1})^{n\kappa} \Gamma_n(\kappa)^{-1} (\operatorname{Paf} h)^{\kappa-2n+1}$$
$$\times (-1)^{[n/2]} D_H^{[n/2](\kappa-2[n/2])} \prod_{i=0}^{n-1} \frac{\zeta(1 + 2i - \kappa)}{\zeta(\kappa - 2i)} \prod_p F_{p,h}\bigl(p^{-\kappa}\bigr).$$

This is equal to $D_h^{\kappa-2n+1} \prod_p F_{p,h}(p^{-\kappa})$ in view of

$$\prod_{i=0}^{n-1} \frac{\zeta(1 + 2i - \kappa)}{\zeta(\kappa - 2i)} = 2^{n(n-\kappa)} \pi^{-n\kappa} (-1)^{n(n-1)/2 - n\kappa/2} \Gamma_n(\kappa).$$

We can complete the proof of Proposition 3.2, employing Proposition 2.1. □

4. Main theorems

For future reference we first let F be an arbitrary number field and H a quaternion algebra central over F. For each finite place v, we write q_v for the cardinality of the residue field of F_v. Let $\mathfrak{S}_H$ denote the set of finite places v of F such that $H_v = H \otimes_F F_v$ is a division algebra. We here collect some information about the local degenerate principal series representation $I_{n,v}(s)$ for $G_n(F_v)$, which is defined over F_v similarly as before.

Proposition 4.1. *(Cf. [3,37].)*

(1) *For any place v of F, if $\Re s = 0$, then $I_{n,v}(s)$ is irreducible.*

(2) *Let $\epsilon \in \{\pm 1\}$. Assume that $v \in \mathfrak{S}_H$. Then $I_{n,v}(\frac{1}{2} + \frac{(1-\epsilon)\pi\sqrt{-1}}{2\log q_v})$ is of length 2 and its Jordan–Hölder sequence*

$$0 \to A_v^n(\epsilon) \to I_{n,v}\left(\frac{1}{2} + \frac{(1-\epsilon)\pi\sqrt{-1}}{2\log q_v}\right) \to B_v^n(\epsilon) \to 0$$

does not split. That is, $A_v^n(\epsilon)$ is a unique irreducible submodule and $B_v^n(\epsilon)$ is a unique irreducible quotient. Both $A_v^n(\epsilon)$ and $B_v^n(\epsilon)$ are unitarizable. Moreover, $A_v^n(\epsilon)$ contains a nonzero $C_{n,v}$-invariant vector if and only if n is even. Furthermore, $A_v^n(\epsilon)$ is of rank n and $B_v^n(\epsilon)$ is of rank $n-1$ in the sense of [12,26].

Proof. Note that $A_v^n(\epsilon)$ (resp. $B_v^n(\epsilon)$) arises via the Weil representation associated to a nondegenerate hermitian form over H_v of dimension n (resp. $n-1$). Thus the last assertion follows easily from a basic calculation based on Lemme on p. 73 of [27]. Other assertions are included in [3,37]. □

From now on we take $F = \mathbb{Q}$. Fix a positive divisor d of D_H. We write Q_d for the set of prime factors of d. Let $f \in S_k(\Gamma_0(d))$ be a primitive form with Hecke L-function:

$$L(f,s) = \prod_{p|d}\left(1-\alpha_p p^{(k-1)/2-s}\right)^{-1} \prod_{p\nmid d}\left(1-\alpha_p p^{(k-1)/2-s}\right)^{-1}\left(1-\alpha_p^{-1} p^{(k-1)/2-s}\right)^{-1}.$$

Put $\epsilon_p = \alpha_p p^{1/2}$ for $p \in Q_d$. Recall that ϵ_p coincides with the negative of the Atkin–Lehner sign of f. For each $h \in T_n^+$ we put

$$A(h) = D_h^{(k-1)/2} \prod_p \widetilde{F}_{p,h}(\alpha_p).$$

We define the function $\mathrm{Lift}_n(f)$ on $\mathfrak{H}_n$ by

$$\mathrm{Lift}_n(f)(Z) = \sum_{h \in T_n^+} A(h)\mathbf{e}\big(\lambda(hZ)\big).$$

This series is absolutely and uniformly convergent on any compact subset of $\mathfrak{H}_n$. This fact can be proven in exactly the same way as in §4 of [14] by using (2.1) and Proposition 2.1.

On the other hand, let $\mathcal{D}_\ell^n$ denote the irreducible lowest weight module of $G_n(\mathbb{R})$ with lowest weight $\det^\ell$. For each place v of $\mathbb{Q}$ we set

$$\Pi_{n,v}(f) = \begin{cases} \mathcal{D}^n_{(k+2n-2)/2} & \text{if } v = \infty, \\ A_p^n(\epsilon_p) & \text{if } v = p \in Q_d, \\ I_{n,p}(s_p) & \text{if } v = p \notin \{\infty\} \cup Q_d. \end{cases}$$

Here s_p is a complex number satisfying $p^{s_p} = \alpha_p$. We consider the restricted tensor product $\Pi_n(f) = \bigotimes_v \Pi_{n,v}(f)$.

Our main theorem is the following:

Theorem 4.2. *Notation being as above, we assume that $d = 1$. Then $\mathrm{Lift}_n(f)$ is a Hecke eigenform in $S_{k+2n-2}(\Gamma_n)$. Moreover, $\Pi_n(f)$ is isomorphic to the cuspidal automorphic representation associated to $\mathrm{Lift}_n(f)$.*

For any nonzero element $h \in T_n$, we put

$$\epsilon(h) = \max\{a \in \mathbb{N} \mid a^{-1}h \in T_n\}.$$

Definition 4.1. Let $M_\kappa^M(\Gamma_2)$ be a subspace of $M_\kappa(\Gamma_2)$ defined as follows: A modular form $F \in M_\kappa(\Gamma_2)$ is an element of $M_\kappa^M(\Gamma_2)$ if there exists a function $c : \mathbb{Z} \to \mathbb{C}$ such that all nonzero $h \in T_2$ satisfy

$$A_F(h) = \sum_{a \in \mathbb{N},\, a | \epsilon(h)} a^{\kappa-1} c\big(a^{-2} D_h\big).$$

Set $S_\kappa^M(\Gamma_2) = M_\kappa^M(\Gamma_2) \cap S_\kappa(\Gamma_2)$.

We shall prove the following result in Section 9.

Theorem 4.3. *Notation being as above, we assume that $n = 2$. Then $\mathrm{Lift}_2(f)$ is a Hecke eigenform in $S_{k+2}^M(\Gamma_2)$. Moreover, $\Pi_2(f)$ is isomorphic to the cuspidal automorphic representation associated to $\mathrm{Lift}_2(f)$. Furthermore, the lifting $f \mapsto \mathrm{Lift}_2(f)$ gives a bijective correspondence (up to a scalar) between Hecke eigenforms in $\bigoplus_{d \geqslant 1, d | D_H} S_k^{\mathrm{new}}(d)$ and those in $S_{k+2}^M(\Gamma_2)$.*

Remark 4.2.

(1) When $n = d = 1$, Remark 3.1(2) shows that $\mathrm{Lift}_1(f) = f$ (cf. Remark 10.1(1)).
(2) In terms of L-function, the relations given in Theorem 4.2 or 4.3 read

$$L^{\mathfrak{S}}\big(s, \mathrm{Lift}_n(f), \mathrm{st}\big) = \prod_{j=1}^{2n} L^{\mathfrak{S}}\left(f, s + \frac{k}{2} + n - j\right)$$

as in the introduction (cf. Appendix A).
(3) The construction of $\mathrm{Lift}_n(f)$ is independent of the choice of α_p by virtue of (2.2). It is perhaps worthwhile to note that for each $h \in T_n^+$ the hth Fourier coefficient of $\mathcal{E}_{k+2n-2}^{(n)}$ is quite similar to that of $\mathrm{Lift}_n(f)$ and this similarity is important for the proofs of Theorems 4.2 and 4.3.

5. Fourier–Jacobi expansions

We use the notation

$$\mathbf{v}(x, y; z) = \left(\begin{array}{cc|cc} \mathbf{1}_m & x & z + \frac{xy^* - yx^*}{2} & y \\ 0 & \mathbf{1}_r & y^* & \mathbf{0}_r \\ \hline & & \mathbf{1}_m & 0 \\ \multicolumn{2}{c|}{\mathbf{0}_{m+r}} & -x^* & \mathbf{1}_r \end{array}\right) \quad \big(x, y \in \mathrm{M}_{mr}(H),\ z \in S_m(H)\big).$$

We define some subgroups of G_{m+r} by

$$Z(\mathbb{Q}) = \big\{\mathbf{v}(0, 0; z) \mid z \in S_m(H)\big\},$$

$$V(\mathbb{Q}) = \big\{\mathbf{v}(x, y; z) \mid x, y \in \mathrm{M}_{mr}(H),\ z \in S_m(H)\big\},$$

$$X(\mathbb{Q}) = \big\{\mathbf{v}(x, 0; 0) \mid x \in \mathrm{M}_{mr}(H)\big\}.$$

We identify $Z(\mathbb{A})$ (resp. $X(\mathbb{A})$) with $S_m(H \otimes_{\mathbb{Q}} \mathbb{A})$ (resp. $\mathrm{M}_{mr}(H \otimes_{\mathbb{Q}} \mathbb{A})$) frequently. We will specialize to the case $m = n-1$ and $r = 1$ in our application to the proof of Theorem 4.2.

Fix $S \in S_m^+(H)$. We regard S as a homomorphism $Z \to \mathbb{G}_a$ by $z \mapsto \lambda(Sz)$, where $\mathbb{G}_a$ is the additive group in one variable. If we put $V_0 = V/\operatorname{Ker} S$, then V_0 is a Heisenberg group with center $Z/\operatorname{Ker} S$ and a natural symplectic structure V/Z. For each place v the Schrödinger representation $\omega_{S,v}$ of $V(\mathbb{Q}_v)$ with central character $z \mapsto \mathbf{e}_v(\lambda(Sz))$ realized on the Schwartz space $\mathcal{S}(X(\mathbb{Q}_v))$ of $X(\mathbb{Q}_v)$ is given by

$$\omega_{S,v}\big(\mathbf{v}(x,y;z)\big)\varphi(t) = \varphi(t+x)\mathbf{e}_v\big(\lambda(Sz) + \lambda\big(2t^*Sy + x^*Sy\big)\big)$$

for $\varphi \in \mathcal{S}(X(\mathbb{Q}_v))$. By the Stone–von Neumann theorem, $\omega_{S,v}$ is a unique irreducible representation of $V(\mathbb{Q}_v)$ on which $Z(\mathbb{Q}_v)$ acts by $z \mapsto \mathbf{e}_v(\lambda(Sz))$.

We identify G_r with a subgroup of G_{m+r} via the embedding $g \mapsto \iota_0(\mathbf{1}_{2m}, g)$ (see Notation). Put $J = G_r \cdot V$. This embedding and the conjugating action give a homomorphism $G_r \hookrightarrow \mathrm{Sp}_{V/Z}$, and Kudla [25] gives an explicit local splitting $G_r(\mathbb{Q}_v) \hookrightarrow \widetilde{\mathrm{Sp}_{V/Z}(\mathbb{Q}_v)}$, where $\widetilde{\mathrm{Sp}_{V/Z}(\mathbb{Q}_v)}$ is the metaplectic extension of $\mathrm{Sp}_{V/Z}(\mathbb{Q}_v)$. The Schrödinger representation $\omega_{S,v}$ of $V(\mathbb{Q}_v)$ extends to the Weil representation of $\widetilde{\mathrm{Sp}_{V/Z}(\mathbb{Q}_v)} \ltimes V(\mathbb{Q}_v)$. The pullback to $G_r(\mathbb{Q}_v)$ of this representation is given by

$$\omega_{S,v}\big(m(a)\big)\varphi(t) = \big|\nu(a)\big|_v^m \varphi(ta),$$
$$\omega_{S,v}\big(n(b)\big)\varphi(t) = \mathbf{e}_v\big(\lambda\big(S[t]b\big)\big)\varphi(t),$$
$$\omega_{S,v}\left(\begin{pmatrix} 0 & -\mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix}\right)\varphi(t) = h_v^{mr} \int\limits_{X(\mathbb{Q}_v)} \varphi(x)\mathbf{e}_v\big(-2\lambda\big(t^*Sx\big)\big)\,dx$$

for $t \in X(\mathbb{Q}_v)$, $a \in \mathrm{GL}_r(H_v)$ and $b \in S_r(H_v)$. Here we take the self-dual Haar measure on $X(\mathbb{Q}_v)$ with respect to the Fourier transform above, and let $h_v = 1$ or -1 according as $H_v \simeq \mathrm{M}_2(\mathbb{Q}_v)$ or not. The global Weil representation ω_S of $J(\mathbb{A})$ on $\mathcal{S}(X(\mathbb{A}))$ is given by the restricted tensor product of $\omega_{S,v}$.

For each $l \in \mathcal{S}(X(\mathbb{A}_\mathrm{f}))$ we define $l' \in \mathcal{S}(X(\mathbb{A}))$ by

$$l'(x) = \varphi^0(x_\infty)l(x_\mathbf{f}), \qquad \varphi^0(x_\infty) = e^{-2\pi\lambda(S[x_\infty])}$$

for $x \in X(\mathbb{A})$, and put

$$\Theta^S(vg;l) = \sum_{\xi \in X(\mathbb{Q})} \omega_S(vg)l'(\xi) \quad \big(v \in V(\mathbb{A}),\ g \in G_r(\mathbb{A})\big).$$

As a general fact, this function is left invariant under $J(\mathbb{Q})$.

Remark 5.1. One can readily verify that

$$\big(\omega_{S,\infty}(\alpha)\varphi^0\big)(x_\infty) = j_{2m}(\alpha,\mathbf{i})^{-1}\mathbf{e}\big(\lambda\big(S[x_\infty]\alpha(\mathbf{i})\big)\big) \quad \big(\alpha \in G_r(\mathbb{R}),\ x_\infty \in X(\mathbb{R})\big).$$

In particular, $\omega_{S,\infty}(w)\varphi^0 = (\det \mathbf{w})^m\varphi^0$ for $w \in C_{r,\infty} \simeq U(2r)$.

Put $\mathfrak{X} = X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. For each $l \in \mathcal{S}(X(\mathbb{A}_\mathrm{f}))$ the theta series on $\mathcal{D} = \mathfrak{H}_r \times \mathfrak{X}$ is defined by

$$\vartheta_l^S(Z,u) = \sum_{\xi \in X(\mathbb{Q})} l(\xi)\mathbf{e}\big(\lambda\big(S[\xi]Z + 2\xi^*Su\big)\big).$$

The relation between the adelic theta functions on $J(\mathbb{A})$ and the classical theta series on $\mathcal{D}$ is made explicit as follows. The group $J(\mathbb{R})$ acts on $\mathcal{D}$ by

$$\beta(Z,u) = \bigl(\alpha Z, u(cZ+d)^{-1} + x(\alpha Z) + y\bigr)$$

for $\beta = v\alpha$ with $v = \mathbf{v}(x,y;z) \in V(\mathbb{R})$ and $\alpha = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in G_r(\mathbb{R})$, and the automorphy factor on $J(\mathbb{R}) \times \mathcal{D}$ is defined by

$$\begin{aligned} j_{\kappa,S}\bigl(\beta,(Z,u)\bigr) &= j_\kappa(\alpha,Z) \\ &\quad \times \mathbf{e}\bigl[-\lambda(Sz) + \lambda\bigl\{\bigl(cS[u] - 2x^*Su\bigr)(cZ+d)^{-1} - S[x](\alpha Z) - x^*Sy\bigr\}\bigr]. \end{aligned}$$

For each $\mathbb{C}$-valued function ϕ on $\mathcal{D}$, we define $\phi|_{\kappa,S}\beta : \mathcal{D} \to \mathbb{C}$ by

$$\phi|_{\kappa,S}\beta(Z,u) = j_{\kappa,S}\bigl(\beta,(Z,u)\bigr)^{-1}\phi\bigl(\beta(Z,u)\bigr).$$

By Remark 5.1, we see that

$$\begin{aligned} \Theta^S(\beta;l) &= \sum_{\xi \in X(\mathbb{Q})} l(\xi)\bigl(\omega_{S,\infty}(\alpha)\varphi^0\bigr)(\xi + x)\mathbf{e}\bigl(\lambda(Sz) + \lambda\bigl(2\xi^*Sy + x^*Sy\bigr)\bigr) \\ &= \sum_{\xi \in X(\mathbb{Q})} l(\xi) j_{2m}(\alpha,\mathbf{i})^{-1}\mathbf{e}\bigl(\lambda(Sz) + \lambda\bigl(S[\xi + x]\alpha(\mathbf{i}) + 2\xi^*Sy + x^*Sy\bigr)\bigr) \\ &= \vartheta_l^S\big|_{2m,S}\beta(\mathbf{i},0). \end{aligned}$$

The left invariance of $\Theta^S(vg;l)$ under $G_r(\mathbb{Q})$ entails the following transformation equation

$$j_{2m}\bigl(\gamma,(Z,u)\bigr)^{-1}\vartheta_l^S\bigl(\gamma(Z,u)\bigr) = \vartheta^S_{\omega_S(\gamma^{-1})l}(Z,u)$$

for all $\gamma \in G_r(\mathbb{Q})$.

For later use we introduce an auxiliary space $\mathcal{W}_{\kappa,S}$ as follows. Let $\mathcal{W}_{\kappa,S}$ be the space of complex valued functions Φ on $J(\mathbb{A})$ such that:

(i) For $z \in Z(\mathbb{A})$, $\gamma \in V(\mathbb{Q})$, $j \in J(\mathbb{A})$ and $w \in C_{r,\infty}$, we have

$$\Phi(z\gamma jw) = \mathbf{e}_{\mathbb{A}}\bigl(\lambda(Sz)\bigr)\Phi(j)(\det \mathbf{w})^{\kappa/2}.$$

(ii) There is a compact open subgroup $K \subset J(\mathbb{A}_\mathrm{f})$ such that Φ is invariant under right translation by K.

(iii) For each $\delta \in G_r(\mathbb{A}_\mathrm{f})$ we define a function Φ_δ on $\mathcal{D}$ via

$$\Phi_\delta|_{\kappa,S}\beta(\mathbf{i},0) = \Phi(\beta\delta), \quad \beta \in J(\mathbb{R}).$$

Then Φ_δ is holomorphic and possesses a Fourier expansion of the form

$$\Phi_\delta(Z,u) = \sum_{B \in X(\mathbb{Q}),\, N \in S_r(H)} c(N,B)\mathbf{e}\bigl(\lambda\bigl(NZ + 2B^*Su\bigr)\bigr),$$

where $c(N,B) = 0$ unless $S_{B,N} = \left(\begin{smallmatrix} S & SB \\ B^*S & N \end{smallmatrix}\right)$ is positive semi-definite.

For $\varPhi \in \mathcal{W}_{\kappa,S}$ and $l \in \mathcal{S}(X(\mathbb{A}_{\mathrm{f}}))$ we put

$$J_l^S(g;\varPhi) = \int\limits_{V(\mathbb{Q})\backslash V(\mathbb{A})} \varPhi(vg)\overline{\Theta^S(vg;l)}\,dv, \quad g \in G_r(\mathbb{A}).$$

Lemma 5.1. *Let $\varPhi \in \mathcal{W}_{\kappa,S}$. Fix $\gamma \in G_r(\mathbb{Q})$ and $g \in G_r(\mathbb{A})$. Then the following conditions are equivalent*:

(a) $\varPhi(\gamma vg) = \varPhi(vg)$ *for all* $v \in V(\mathbb{A})$;
(b) $J_l^S(\gamma g;\varPhi) = J_l^S(g;\varPhi)$ *for all* $l \in \mathcal{S}(X(\mathbb{A}_{\mathrm{f}}))$.

Proof. There is no difficulty in proving that (a) implies (b). We may assume that $v \in V(\mathbb{R})$ in proving (a). Put $Z = g_\infty(\mathbf{i})$. Define a function $\varPsi$ on $\mathfrak{X}$ by

$$\varPsi(u) = \varPhi_{\gamma_{\mathbf{f}} g_{\mathbf{f}}}|_{\kappa,S}\gamma(Z,u).$$

We claim that $\varPsi$ has the following property: there exists a lattice $L \subset X(\mathbb{Q})$ such that the equalities

$$\mathbf{e}\big(\lambda\big(2\xi^* Su + S[\xi]Z + \xi^* S\eta\big)\big)\varPsi(u + \xi Z + \eta) = \varPsi(u)$$

hold for all $\xi,\ \eta \in L$. Indeed, if we let $v^\gamma = \gamma^{-1}v\gamma$ and $(Z,u) = v^\gamma g_\infty(\mathbf{i},0)$, then a straightforward comparison of the definition of the space $\mathcal{W}_{\kappa,S}$ shows that the left-hand side is equal to

$$\begin{aligned}
\varPhi_{\gamma_{\mathbf{f}} g_{\mathbf{f}}}|_{\kappa,S}\gamma\mathbf{v}(\xi,\eta;0)(Z,u) &= j_{\kappa,S}\big(v^\gamma g_\infty,(\mathbf{i},0)\big)\varPhi_{\gamma_{\mathbf{f}} g_{\mathbf{f}}}|_{\kappa,S}\gamma\mathbf{v}(\xi,\eta;0)v^\gamma g_\infty(\mathbf{i},0) \\
&= j_{\kappa,S}\big(v^\gamma g_\infty,(\mathbf{i},0)\big)\varPhi\big(\gamma\mathbf{v}(\xi,\eta;0)v^\gamma g\big) \\
&= j_{\kappa,S}\big(v^\gamma g_\infty,(\mathbf{i},0)\big)\varPhi\big(\gamma v^\gamma g\big) = \varPsi(u).
\end{aligned}$$

Put

$$V_L = \big\{\mathbf{v}(\xi,\eta;0) \bigm| \xi,\eta \in L\backslash X(\mathbb{R})\big\}, \qquad L_Z = \{\xi Z + \eta \mid \xi,\eta \in L\}.$$

Provided that L is a sufficiently small lattice of $X(\mathbb{Q})$, we see that

$$\begin{aligned}
J_l^S(\gamma g;\varPhi) &= \int\limits_{V(\mathbb{Q})\backslash V(\mathbb{A})} \varPhi(v\gamma g)\overline{\Theta^S(v\gamma g;l)}\,dv \\
&= \int\limits_{V(\mathbb{Q})\backslash V(\mathbb{A})} \varPhi\big(\gamma v^\gamma g\big)\overline{\Theta^S\big(v^\gamma g;l\big)}\,dv \\
&= \int\limits_{V_L} \varPhi_{\gamma_{\mathbf{f}} g_{\mathbf{f}}}|_{\kappa,S}\gamma v g_\infty(\mathbf{i},0)\overline{\vartheta^S_{\omega_S(g_{\mathbf{f}})l}|_{2m,S} v g_\infty(\mathbf{i},0)}\,dv \\
&= \frac{(\mathrm{Paf}\,\Im Z)^{2m}}{j_{\kappa-2m}(g_\infty,\mathbf{i})} \int\limits_{L_Z\backslash\mathfrak{X}} \varPsi(u)\overline{\vartheta^S_{\omega_S(g_{\mathbf{f}})l}(Z,u)}e^{-4\pi\lambda((\Im Z)^{-1}S[\Im u])}\,du,
\end{aligned}$$

where du is the measure on $L_Z\backslash\mathfrak{X}$ normalized to have total volume 1. Now our lemma is an easy consequence of the fact that any holomorphic function on $\mathfrak{X}$ with the above property equals $u \mapsto \vartheta_l^S(Z,u)$ for some $l \in \mathcal{S}(X(\mathbb{A}_{\mathrm{f}}))$. $\square$

Definition 5.2. Let $\mathcal{G}$ be a smooth function on $P_{m+r}(\mathbb{Q})\backslash G_{m+r}(\mathbb{A})$. For each $S \in S_m(H)$ the Sth Fourier–Jacobi coefficient of $\mathcal{G}$ is a function on $V(\mathbb{Q})\backslash J(\mathbb{A})$ defined by

$$\mathcal{G}_S(vg) = \int\limits_{Z(\mathbb{Q})\backslash Z(\mathbb{A})} \mathcal{G}(zvg)\mathbf{e}_{\mathbb{A}}\big(-\lambda(Sz)\big)\,dz \quad \big(v \in V(\mathbb{A}),\ g \in G_r(\mathbb{A})\big).$$

The main point to notice here is that $\mathcal{G}$ is not assumed to be left invariant under $G_{m+r}(\mathbb{Q})$. Now let $S \in S_m^+(H)$. If $\mathcal{G}_S \in \mathcal{W}_{\kappa,S}$, then we define a function on $G_r(\mathbb{A})$ by $\mathrm{FJ}_l^S(g;\mathcal{G}) = J_l^S(g;\mathcal{G}_S)$ for each $l \in \mathcal{S}(X(\mathbb{A}_{\mathrm{f}}))$. Moreover, for each $\delta \in G_r(\mathbb{A}_{\mathrm{f}})$, we define a function $\mathrm{FJ}_{l,\delta}^S(\mathcal{G}) : \mathfrak{H}_r \to \mathbb{C}$ by

$$\mathrm{FJ}_{l,\delta}^S(\mathcal{G})|_{\kappa-2m}\alpha(\mathbf{i}) = \mathrm{FJ}_l^S(\alpha\delta;\mathcal{G}), \quad \alpha \in G_r(\mathbb{R}).$$

This is well defined and is equal to

$$\mathrm{FJ}_{l,\delta}^S(\mathcal{G})(Z) = (\operatorname{Paf}\Im Z)^{2m} \int\limits_{L_Z\backslash\mathfrak{X}} (\mathcal{G}_S)_\delta(Z,u)\overline{\vartheta^S_{\omega_S(\delta)l}(Z,u)}e^{-4\pi\lambda((\Im Z)^{-1}S[\Im u])}\,du$$

by the proof of Lemma 5.1, where L is a sufficiently small lattice of $X(\mathbb{Q})$. If we write

$$(\mathcal{G}_S)_\delta(Z,u) = \sum_{B\in X(\mathbb{Q}),\ N\in S_r(H)} c(N,B)\mathbf{e}\big(\lambda\big(NZ + 2B^*Su\big)\big)$$

and if L is sufficiently small, then

$$c\big(N + S[B+\eta] - S[B], B+\eta\big) = c(N,B), \qquad \omega_S(\delta)l(\xi+\eta) = \omega_S(\delta)l(\xi)$$

for $\eta \in L$. Since the integration with respect to $\Re u$ vanishes unless $B = \xi$,

$$\begin{aligned}
\mathrm{FJ}_{l,\delta}^S(\mathcal{G})(Z) &= (\operatorname{Paf}\Im Z)^{2m} \int\limits_{L_Z\backslash\mathfrak{X}} \sum_{B,\xi\in X(\mathbb{Q}),\ N\in S_r(H)} \overline{\omega_S(\delta)l(\xi)}c(N,B) \\
&\quad \times \mathbf{e}\big(\lambda\big(NZ - S[\xi]\overline{Z}\big) + 2\lambda\big(B^*Su - \xi^*S\bar{u}\big)\big)e^{-4\pi\lambda((\Im Z)^{-1}S[\Im u])}\,du \\
&= (\operatorname{Paf}\Im Z)^{2m} \int\limits_{L(\Im Z)\backslash X(\mathbb{R})} \sum_{\xi\in X(\mathbb{Q}),\ N\in S_r(\mathbb{Q})} \overline{\omega_S(\delta)l(\xi)}c(N,\xi) \\
&\quad \times \mathbf{e}\big(\lambda\big(\big(N - S[\xi]\big)Z\big)\big)e^{-4\pi\lambda((\Im Z)^{-1}S[\Im u+\xi\Im Z])}\,d\Im u \\
&= C \sum_{\xi\in X(\mathbb{Q})/L,\ N\in S_r(H)} \overline{\omega_S(\delta)l(\xi)}c(N,\xi)\mathbf{e}\big(\lambda\big(\big(N - S[\xi]\big)Z\big)\big)
\end{aligned}$$

with

$$C = (\operatorname{Paf}\Im Z)^{2m} \int\limits_{X(\mathbb{R})} e^{-4\pi\lambda((\Im Z)^{-1}S[\Im u])}\,d\Im u.$$

Note that C is a constant because of our normalization of the measure du.

Lemma 5.2. *(Cf. [13].) Let $S \in S_m^+(H)$ and $l \in \mathcal{S}(X(\mathbb{A}_f))$. If $\Phi \in I_{m+r}(s,\chi)$ and $\Re s \gg 0$, then $\mathrm{FJ}_l^S(g; E_\Phi)$ is an Eisenstein series on $G_r(\mathbb{A})$ associated to some element of $I_r(s,\chi)$.*

Proof. The proof is totally analogous to that of [13, Theorem 3.2]. $\square$

For later use we give definitions of Jacobi forms and Fourier–Jacobi coefficients of modular forms in classical language. Let $S \in T_m^+$. Put

$$\Gamma_{m,r} = J(\mathbb{Q}) \cap \mathrm{GL}_{2(m+r)}(R).$$

Definition 5.3. A Jacobi form (resp. Jacobi cusp form) ϕ of weight κ and index S is a holomorphic function on $\mathcal{D}$ satisfying the following conditions:

- $\phi|_{\kappa,S}\gamma = \phi$ for every $\gamma \in \Gamma_{m,r}$;
- For each $\gamma \in J(\mathbb{Q})$, $\phi|_{\kappa,S}\gamma$ has a Fourier expansion of the form

$$\phi|_{\kappa,S}\gamma(Z,u) = \sum_{B\in X(\mathbb{Q}),\, N\in S_r(H)} c(N,B)\mathbf{e}\bigl(\lambda\bigl(NZ + 2B^*Su\bigr)\bigr),$$

where $c(N,B) = 0$ unless $S_{B,N} = \begin{pmatrix} S & SB \\ B^*S & N \end{pmatrix}$ is positive semi-definite (resp. positive definite).

We denote by $J_{\kappa,S}(\Gamma_{m,r})$ (resp. $J_{\kappa,S}^{\mathrm{cusp}}(\Gamma_{m,r})$) the space of Jacobi forms (resp. Jacobi cusp forms) of weight κ and index S.

Put $\Xi(S) = S^{-1}(\mathrm{M}_{mr}(\widetilde{R}))/\mathrm{M}_{mr}(R)$. For $\xi \in \Xi(S)$ let $l_\xi \in \mathcal{S}(X(\mathbb{A}_f))$ be the characteristic function of $\xi + \prod_p \mathrm{M}_{mr}(R_p)$ (for the definition of $\widetilde{R}$, see the proof of Lemma 2.4). A holomorphic function $\phi : \mathcal{D} \to \mathbb{C}$ satisfying $\phi|_{\kappa,S}\gamma = \phi$ for all $\gamma \in V(\mathbb{Q}) \cap \mathrm{GL}_{2(m+r)}(R)$ has an expansion of the form

$$\phi(Z,u) = \sum_{\xi\in\Xi(S)} \phi_\xi(Z)\vartheta_{l_\xi}^S(Z,u),$$
$$\phi_\xi(Z) = \sum_{N\in S_r(H)} c(N,\xi)\mathbf{e}\bigl(\lambda\bigl(\bigl(N - S[\xi]\bigr)Z\bigr)\bigr).$$

This is a simple variant of Lemma 5.1 in the classical setting.

The Sth Fourier–Jacobi coefficient F_S of $F \in M_\kappa(\Gamma_{m+r})$ is defined by

$$F_S(Z,u) = \sum_{B\in X(\mathbb{Q}),\, N\in S_r(H)} A_F(S_{B,N})\mathbf{e}\bigl(\lambda\bigl(NZ + 2B^*Su\bigr)\bigr).$$

It is immediate that $F_S \in J_{\kappa,S}(\Gamma_{m,r})$.

6. Proof of main theorem

We begin by fixing some notation and reviewing Ikeda's results of [15]. Put $\mathcal{K} = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$. Let (u,V) be a finite dimensional continuous representation of $\mathcal{K}$. A V-valued modular form $\vec{g}$ of weight k' with type u is a V-valued holomorphic function on $\mathfrak{H}_1$ which satisfies the following conditions:

- $\vec{g}(\gamma\tau) = j_{k'}(\gamma, \tau)u(\gamma)\vec{g}(\tau)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$;
- $\vec{g}$ has a Fourier expansion of the form

$$\vec{g}(\tau) = \sum_{N=0}^{\infty} \vec{c}(N)q^{N/M}$$

for some positive integer M.

For each rational prime p let $\mathcal{R}_p$ be a copy of the reciprocal Laurent polynomial ring $\mathbb{C}[X_p + X_p^{-1}]$. Put $\mathcal{R} = \otimes'_p \mathcal{R}_p$. For each sequence $a_2, a_3, \ldots, a_p, \ldots$ of nonzero complex numbers indexed by prime numbers, the value of $\Phi \in \mathcal{R}$ at $(X_2, X_3, \ldots, X_p, \ldots) = (a_2, a_3, \ldots, a_p, \ldots)$ is denoted by $\Phi(\{a_p\})$.

When h is a modular form of weight k' for some congruence subgroup Γ, let $\mathcal{V}(h)$ be the $\mathbb{C}$-vector space spanned by $\{h|_{k'}\gamma \mid \gamma \in \mathrm{GL}_2^+(\mathbb{Q})\}$. Following Ikeda [15], we define a compatible family of Eisenstein series as follows (cf. Remark 3.1(2)). When k' extends over sufficiently large even integers, a compatible family of Eisenstein series $\{g_{k'}\}_{k'}$ is a family of modular forms

$$g_{k'}(\tau) = b_{k'}(0) + \sum_{N \in \mathbb{Q}_+^\times} N^{(k'-1)/2} b_{k'}(N) q^N$$

satisfying the following conditions:

(i) $g_{k'} \in \mathcal{V}(E^1_{k'}(\tau))$ for each k', where $E^1_{k'}(\tau) := E^1_{k'}(\tau, 0)$;
(ii) for each $N \in \mathbb{Q}_+^\times$ there is an element $\Phi_N \in \mathcal{R}$ such that

$$b_{k'}(N) = \Phi_N\big(\big\{p^{(k'-1)/2}\big\}\big);$$

(iii) there is a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $g_{k'} \in M_{k'}(\Gamma)$ for each k'.

Then Ikeda obtained the following:

Lemma 6.1. *(Cf. [15, Lemma 10.2].) Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform with Satake p-parameters $\{\alpha_p, \alpha_p^{-1}\}$. Assume that there is a finite dimensional representation $(u, \mathbb{C}^d)$ of $\mathcal{K}$ and $\vec{\Phi}_N = {}^t(\Phi_{1,N}, \ldots, \Phi_{d,N})$ $(N \in \mathbb{Q}_+^\times)$ satisfying the following conditions*:

(a) *there exists a vector valued modular form*

$$\vec{g}_{k'}(\tau) = \vec{b}_{k'}(0) + \sum_{N \in \mathbb{Q}_+^\times} N^{(k'-1)/2} \vec{b}_{k'}(N) q^N$$

of weight k' with type u for each sufficiently large even integers k';
(b) *for each i $(1 \leqslant i \leqslant d)$ the ith component of $\vec{g}_{k'}$ is a compatible family of Eisenstein series such that*

$$b_{i,k'}(N) = \Phi_{i,N}\big(\big\{p^{(k'-1)/2}\big\}\big),$$

where $\vec{b}_{k'}(N) = {}^t(b_{1,k'}(N), \ldots, b_{d,k'}(N))$.

Put

$$\vec{h}(\tau) = \sum_{N \in \mathbb{Q}_+^\times} N^{(k-1)/2} \vec{\Phi}_N\big(\{\alpha_p\}\big) q^N.$$

Then $\vec{h}$ is a vector valued modular form of weight k with type u.

We now turn to the proof of Theorem 4.2. We may assume that $n \geqslant 2$ by Remark 4.2(1). For ease of notations, we put $F = \mathrm{Lift}_n(f)$ and $\kappa = k + 2n - 2$.

First of all, F is not identically zero as Proposition 2.1 implies that $A(h) = 1$ whenever $h \in T_n^+$ satisfies $D_h = 1$. Lemma 2.4 ensures the existence of such h.

Next, it is easy to see that $F|_\kappa \gamma = F$ for $\gamma \in P_n(\mathbb{Q}) \cap \Gamma_n$. Thus (1.2) allows us to define a function $\mathcal{F}$ on $G_n(\mathbb{A})$ by

$$\mathcal{F}(\rho \alpha w) = F\big(\alpha(\mathbf{i})\big) j_\kappa(\alpha w_\infty, \mathbf{i})^{-1}$$

for $\rho \in P_n(\mathbb{Q})$, $\alpha \in G_n(\mathbb{R})$ and $w \in C_n$. The function $\mathcal{F}$ satisfies

$$\mathcal{F}(\rho g w) = \mathcal{F}(g)(\det \mathbf{w}_\infty)^{\kappa/2}$$

for every $\rho \in P_n(\mathbb{Q})$, $g \in G_n(\mathbb{A})$ and $w \in C_n$.

Recall that suitable generators of the Siegel and hermitian modular groups play an important role in Ikeda's constructions of liftings in [14,15], as it did in the proof of the Saito–Kurokawa conjecture. Unfortunately, we do not yet know such generators for quaternion modular groups except for the Hurwitz quaternion case (see Conjecture 10.4), which is in fact a main technical problem involved in our proof. To sidestep this difficulty, we will work in adelic form, namely, our goal in this section is to prove the following lemma (see Notation for the definition of the embedding $\iota_0 : G_{n-1} \times G_1 \hookrightarrow G_n$).

Lemma 6.2. *$\mathcal{F}$ is left invariant under $\iota_0(\mathbf{1}_{2n-2}, \gamma)$ for any $\gamma \in G_1(\mathbb{Q})$.*

Once this lemma is established, it is clear that $\mathcal{F} \in \mathfrak{S}_\kappa^n$ and hence $F \in S_\kappa(\Gamma_n)$ by Remark 1.3. We shall determine the local components of F in Appendix A.

We will prove Lemma 6.2 in several steps. From now on the group $G_1(\mathbb{A})$ will be viewed as a subgroup of $G_n(\mathbb{A})$ via ι_0. By an Iwasawa decomposition of $G_n(\mathbb{A})$ relative to the Levi subgroup $\iota_0(m(Q_{n-1}), G_1)$, it suffices to prove that $\mathcal{F}(\gamma x) = \mathcal{F}(x)$ for all $\gamma \in G_1(\mathbb{Q})$ and elements x of the form $j\Delta$ for $j \in J(\mathbb{A})$ and $\Delta = \iota_0(m(c), \mathbf{1}_2)$ with $m(c) \in Q_{n-1}(\mathbb{A})$. Fix c and define a function $\mathcal{G} : G_n(\mathbb{A}) \to \mathbb{C}$ by $\mathcal{G}(g) = \mathcal{F}(g\Delta)$.

We should prove the following lemma so as to invoke Lemma 5.1.

Lemma 6.3.

(1) *$\mathcal{G}_S = 0$ unless $S \in S_{n-1}^+(H)$.*
(2) *$\mathcal{G}_S$ is an element of $\mathcal{W}_{\kappa,S}$ for any $S \in S_{n-1}^+(H)$.*

Proof. Fix $\delta \in G_1(\mathbb{A}_\mathrm{f})$. Let $\beta = v\alpha$ with $v = \mathbf{v}(x, y; z) \in V(\mathbb{R})$ and $\alpha \in G_1(\mathbb{R})$. Set $(\tau, u) = \beta(\sqrt{-1}, 0)$. Writing $\beta\delta\Delta$ as $\rho x w \in P_n(\mathbb{Q}) G_n(\mathbb{R}) \prod_p C_{n,p}$, where we may assume that $\rho = n(b)m(a)$ is independent of β, we obtain

$$x(\mathbf{i}) = \rho^{-1}\beta\Delta_\infty(\mathbf{i}) = a^{-1}\left(\begin{pmatrix} \sqrt{-1}c_\infty c_\infty^* + x\tau x^* + \frac{xy^*+yx^*}{2} + z & u \\ u^* & \tau \end{pmatrix} - b\right)(a^{-1})^*,$$

$$j_\kappa(x, \mathbf{i}) = j_\kappa\big(\rho^{-1}\beta\Delta_\infty, \mathbf{i}\big) = \nu(a)^{\kappa/2}\nu(c_\infty)^{-\kappa/2} j_\kappa(\alpha, \mathbf{i}).$$

Then it follows that

$$(\mathcal{G}_S)_\delta(\tau,u) = C_1 \sum_{B\in X(\mathbb{Q}),\, N\in\mathbb{Q}} A\bigl(S_{B,N}[a]\bigr)\mathbf{e}\bigl(-\lambda(S_{B,N}b)\bigr)q^N\mathbf{e}\bigl(2\lambda\bigl(B^*Su\bigr)\bigr),$$
$$C_1 = \nu(a)^{-\kappa/2}\nu(c_\infty)^{\kappa/2}e^{-2\pi\lambda(S[c_\infty])}.$$

Our lemma is now immediate. □

Proof of Lemma 6.2. Fix $S\in S^+_{n-1}(H)$. By Lemmas 5.1 and 6.3, our task is to show that

$$\mathrm{FJ}^S_l(\gamma\alpha\delta;\mathcal{G}) = \mathrm{FJ}^S_l(\alpha\delta;\mathcal{G})$$

for all $\gamma\in G_1(\mathbb{Q})$, $\alpha\in G_1(\mathbb{R})$, $\delta\in G_1(\mathbb{A}_\mathrm{f})$ and $l\in\mathcal{S}(X(\mathbb{A}_\mathrm{f}))$. There is a finite subset Σ of $Q_1(\mathbb{A}_\mathrm{f})$ such that $G_1(\mathbb{A}) = \sqcup_{\sigma\in\Sigma}G_1(\mathbb{Q})G_1(\mathbb{R})\sigma C_1$ by (1.1). Since

$$\mathrm{FJ}^S_l(\alpha\delta w;\mathcal{G}) = \mathrm{FJ}^S_{\omega_S(w)l}(\alpha\delta;\mathcal{G})$$

for $w\in\prod_p C_{1,p}$, we may assume that $\delta = m(d)$ for some $d=(d_p)\in \mathrm{GL}_1(H\otimes_\mathbb{Q}\mathbb{A}_\mathrm{f})$. Choose $t\in\mathbb{Q}^\times_+$ so that $t\nu(d_p)\in\mathbb{Z}_p^\times$ for every prime number p. Put $\varrho = \mathrm{diag}[1,t]$. Since $R' = \bigcap_p(H\cap d_pR_pd_p^{-1})$ is another maximal order of H, we deduce from (1.4) that

$$\begin{aligned} G_1(\mathbb{Q})\cap\left(\delta\prod_p \mathrm{GL}_2(R_p)\delta^{-1}\right) &= G_1(\mathbb{Q})\cap\left(\varrho\prod_p\mathrm{GL}_2\bigl(d_pR_pd_p^{-1}\bigr)\varrho^{-1}\right)\\ &= \varrho\bigl(G_1(\mathbb{Q})\cap\mathrm{GL}_2\bigl(R'\bigr)\bigr)\varrho^{-1}\\ &= \bigl\{\varepsilon\bigm|\varepsilon\in R'^\times\bigr\}\cdot\varrho\mathrm{SL}_2(\mathbb{Z})\varrho^{-1}. \end{aligned}$$

Put $\gamma^\varrho = \varrho\gamma\varrho^{-1}$ and $\omega'_S(\gamma) = \omega_S(\delta^{-1}\gamma^\varrho\delta)$ for $\gamma\in\mathcal{K}$. Fix $l\in\mathcal{S}(X(\mathbb{A}_\mathrm{f}))$. Let V be the $\mathbb{C}$-vector space generated by $\{\omega'_S(\gamma)l\mid\gamma\in\mathcal{K}\}$. Taking a $\mathcal{K}$-invariant nondegenerate hermitian inner product on V, we regard the conjugate linear form $l\mapsto\vec{h}(\tau)(l) = \mathrm{FJ}^S_{l,\delta}(\mathcal{G})(\tau/t)$ as an element of V.

Now it is enough to prove that

$$\vec{h}(\gamma\tau) = j_k(\gamma,\tau)\omega'_S(\gamma)\vec{h}(\tau)$$

for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, i.e., $\vec{h}$ is a V-valued modular form of wight k with type ω'_S. Indeed, if this is so, then letting $\alpha\in G_1(\mathbb{R})$ and $\tau=\alpha(\mathbf{i})$, we have

$$\begin{aligned} \mathrm{FJ}^S_l\bigl(\gamma^\rho\alpha\delta;\mathcal{G}\bigr) &= \mathrm{FJ}^S_{\omega'_S(\gamma)l,\delta}(\mathcal{G})|_k\gamma^\rho\alpha(\mathbf{i})\\ &= j_k(\alpha,\mathbf{i})^{-1}j_k\bigl(\gamma^\varrho,\tau\bigr)^{-1}\vec{h}\bigl(\gamma(t\tau)\bigr)\bigl(\omega'_S(\gamma)l\bigr)\\ &= j_k(\alpha,\mathbf{i})^{-1}j_k\bigl(\gamma^\varrho,\tau\bigr)^{-1}j_k(\gamma,t\tau)\vec{h}(t\tau)(l) = \mathrm{FJ}^S_l(\alpha\delta;\mathcal{G}). \end{aligned}$$

Since $\mathrm{FJ}^S_l(g;\mathcal{G})$ is left invariant under $P_1(\mathbb{Q})$ by construction, the claimed result follows at once in view of $G_1(\mathbb{Q}) = P_1(\mathbb{Q})\cdot\varrho\mathrm{SL}_2(\mathbb{Z})\varrho^{-1}$.

Let $\kappa' = k'+2n-2$. We define $\mathcal{E}_{\kappa'}$ and $\vec{g}_{k'}$, replacing F with $\mathcal{E}^{(n)}_{\kappa'}$ in the definition of $\mathcal{G}$ and $\vec{h}$ respectively. Keeping track of the argument above, we see that $\vec{g}_{k'}$ is a V-valued modular form of weight k' with type ω'_S. Lemma 6.1 in conjunction with the following lemma completes the proof of Lemma 6.2. □

Lemma 6.4. *Fix $l \in \mathcal{S}(X(\mathbb{A}_f))$ and $\delta \in G_1(\mathbb{A}_f)$. Put $k' = \kappa' - 2n + 2$. Then there is $D \in \mathbb{N}$ such that the sequence $\{D^{-(k'-1)/2}\,\mathrm{FJ}^S_{l,\delta}(\mathcal{E}_{\kappa'})\}_{k'}$ makes up a compatible family of Eisenstein series.*

Proof. The condition (iii) obviously holds. The Eisenstein series on $G_1(\mathbb{A})$ factor through those on $\mathrm{GL}_2(\mathbb{A})$ by (1.1), and hence Lemma 5.2 tells us that $\mathrm{FJ}^S_{l,\delta}(\mathcal{E}_{\kappa'})$ satisfies (i). It now remains to show that $\mathrm{FJ}^S_{l,\delta}(\mathcal{E}_{\kappa'})$ satisfies (ii).

We write $B(h)$ for the hth Fourier coefficient of $\mathcal{E}^{(n)}_{\kappa'}$, which is calculated in Proposition 3.2. Notation being as in the proof of Lemma 6.3, if L is a sufficiently small lattice of $X(\mathbb{Q})$, then there is a constant C' such that

$$\mathrm{FJ}^S_{l,\delta}(\mathcal{E}_{\kappa'})(\tau) = C' \sum_{\xi \in X(\mathbb{Q}/L),\, N \in \mathbb{Q}} \overline{\omega_S(\delta)l(\xi)}\, B\big(S_{\xi,N}[a]\big)\mathbf{e}\big(-\lambda(S_{\xi,N}b)\big)q^{N-S[\xi]}$$

by the discussion after Definition 5.2. Note that C' does not depend on κ'. By Proposition 3.2, $D = \nu(a)D_H^{[n/2]}\operatorname{Paf} S$ works. $\square$

7. A Maass space on the quaternion half-space of degree 2

In what follows, we exclusively consider the case $n = 2$. To make our formulas short, we write $J_\kappa = J_{\kappa,1}(\Gamma_{1,1})$, $J^{\mathrm{cusp}}_\kappa = J^{\mathrm{cusp}}_{\kappa,1}(\Gamma_{1,1})$, $\Xi = \Xi(1)$ and $\vartheta_\xi = \vartheta^1_{l_\xi}$ succinctly, where l_ξ denotes the characteristic function of $\xi + \prod_p R_p$ for $\xi \in \Xi$.

Definition 7.1. Let J^M_κ be the subspace of J_κ consisting of all functions ϕ which possesses a Fourier expansion of the form

$$\phi(\tau,u) = \sum_{\xi \in \widetilde{R},\, N \in \mathbb{N},\, \nu(\xi) \leqslant N} c\big(D_H\big(N - \nu(\xi)\big)\big)q^N\mathbf{e}\big(2\lambda(\xi u)\big) \quad (\tau \in \mathfrak{H}_1,\ u \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})$$

for some function $c : \mathbb{N} \cup \{0\} \to \mathbb{C}$. Put $J^{\mathrm{cusp},M}_\kappa = J^M_\kappa \cap J^{\mathrm{cusp}}_\kappa$.

Recall that $\phi \in J_\kappa$ can be expressed as a sum

$$\phi(\tau,u) = \sum_{\xi \in \Xi} \phi_\xi(\tau)\vartheta_\xi(\tau,u), \qquad \phi_\xi(\tau) = \sum_N c(N,\xi)q^{N-\nu(\xi)}.$$

Put $\Xi(p) = \widetilde{R}_p/R_p$. We frequently identify Ξ with $\bigoplus_p \Xi(p)$ and $\mathbb{Z}/D_H\mathbb{Z}$ with $\bigoplus_{p|D_H} \mathbb{Z}/p\mathbb{Z}$ in an obvious way. Define the map $\rho : \Xi \to \mathbb{Z}/D_H\mathbb{Z}$ by $\rho(\xi) = (p\nu(\xi_p))_p$, using these identifications. We readily see that $\phi \in J^M_\kappa$ if and only if ϕ_ξ depends only on $\rho(\xi)$, which allows us to represent ϕ_ξ by $\phi_{\rho(\xi)}$ if $\phi \in J^M_\kappa$. We shall prove the following proposition in Section 8 (see Section 4 for the definition of the space $S^M_\kappa(\Gamma_2)$).

Proposition 7.1. *The $\mathbb{C}$-linear map*

$$\begin{aligned} \jmath : &\sum_{\xi \in \widetilde{R},\, N \in \mathbb{N},\, \nu(\xi) < N} c\big(D_H\big(N - \nu(\xi)\big)\big)q^N\mathbf{e}\big(2\lambda(\xi u)\big) \\ &\mapsto \sum_{h \in T_2^+} \sum_{a \in \mathbb{N},\, a|\epsilon(h)} a^{\kappa-1}c\big(a^{-2}D_h\big)\mathbf{e}\big(\lambda(hZ)\big) \end{aligned}$$

gives an isomorphism of $J^{\mathrm{cusp},M}_\kappa$ onto $S^M_\kappa(\Gamma_2)$.

Remark 7.2. Proposition 7.1 was proven by Krieg [23] under the assumptions (I) and (II). We should note that he actually showed a more general isomorphism from J_κ^M onto $M_\kappa^M(\Gamma_2)$. Unfortunately, his proof does not apply to the general case again by the lack of the knowledge of generators of Γ_2. We get around the problem by connecting the map $\jmath$ with the theta lifting associated to the quadratic form of signature $(2, 6)$.

Let D be a square-free positive integer. Denote the set of all prime factors of D by Q_D. For each $p \in Q_D$ the map $\mathrm{Tr}_p^D : M_k(\Gamma_0(D)) \to M_k(\Gamma_0(p^{-1}D))$ is defined by setting

$$\mathrm{Tr}_p^D(f) = \sum_{\alpha \in \Gamma_0(D)\backslash\Gamma_0(p^{-1}D)} f|_k\alpha.$$

Put

$$M_k^+(D) = \{ f \in M_k(\Gamma_0(D)) \mid \mathrm{Tr}_p^D(f) = 0 \text{ for } p \in Q_D \},$$
$$S_k^+(D) = M_k^+(D) \cap S_k(\Gamma_0(D)).$$

Let $T(p)$ denote the usual Hecke operator acting on the space of modular forms of weight k. We define the map $Q(p) : M_k(\Gamma_0(d)) \to M_k(\Gamma_0(pd))$ by

$$\begin{aligned} g|Q(p) &= g|T(p) - (p+1)p^{k/2-1} g|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{m \in \mathbb{N}\cup\{0\}} \bigl(c_g(pm) - p^k c_g(p^{-1}m)\bigr) q^m \end{aligned}$$

when p and d are coprime. Assume that d divides D. For $g \in M_k(\Gamma_0(d))$, we define $g^* \in M_k(\Gamma_0(D))$ by

$$g^* = g| \prod_{p|d^{-1}D} Q(p).$$

Lemma 7.2. *Let $D \in \mathbb{N}$ be square-free. For given $g \in S_k^{\mathrm{new}}(d)$, the function g^* belongs to $S_k^+(D)$. Moreover, the map $g \mapsto g^*$ yields a $\mathbb{C}$-linear isomorphism*

$$\bigoplus_{d \geqslant 1, d|D} S_k^{\mathrm{new}}(d) \simeq S_k^+(D).$$

Proof. The proof is straightforward and is omitted. □

Proposition 7.3. *For any $\phi \in J_\kappa$ we define $\sigma(\phi) : \mathfrak{H}_1 \to \mathbb{C}$ by $\sigma(\phi) = \phi_0$. Then $\sigma(\phi) \in M_{\kappa-2}^+(D_H)$. Moreover, the mapping $\phi \mapsto \sigma(\phi)$ induces $\mathbb{C}$-linear isomorphisms $J_\kappa^M \simeq M_{\kappa-2}^+(D_H)$ and $J_\kappa^{\mathrm{cusp},M} \simeq S_{\kappa-2}^+(D_H)$.*

Proof. Put $k = \kappa - 2$. Let V be the subspace of $\mathcal{S}(H \otimes_{\mathbb{Q}} \mathbb{A}_{\mathrm{f}})$ consisting of functions l which are 0 outside $\prod_p \widetilde{R}_p$ and constant on cosets modulo $\prod_p R_p$ in $\prod_p \widetilde{R}_p$. It is easy to see that the Weil representation induces a unitary representation u of $\mathcal{K}$ on V by $u(\gamma)l = \omega_1(\gamma)l$ for $l \in V$ and $\gamma \in \mathcal{K}$. Note that $\{l_\xi\}_{\xi \in \Xi}$ form an orthonormal basis of V.

For $\phi \in J_\kappa$ we define the function $\vec{\phi} : \mathfrak{H}_1 \to V$ by $\langle \vec{\phi}, l_\xi \rangle = \phi_\xi$. As in the proof of Lemma 6.2, we can see that $\vec{\phi}$ is a V-valued modular form of weight k and type u. Since $u(\gamma)l_0 = l_0$ for $\gamma \in \Gamma_0(D_H)$, we have $\sigma(\phi) \in M_k(\Gamma_0(D_H))$.

Put $\mathbf{a}(i) = \prod_{p|D_H} \mathbf{a}_p(i)$, where $\mathbf{a}_p(i) = 1$ or $p+1$ according as i is divisible by p or not. Note that $\mathbf{a}(i) = \#\rho^{-1}(i)$. For each prime factor p of D_H we choose an element $\gamma_p \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_p \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & (\bmod p^2), \\ \mathbf{1}_2 & (\bmod (p^{-1}D_H)^2). \end{cases}$$

Observe that

$$u(\gamma_p)l_\xi = -p^{-1} \sum_{\eta \in \Xi(p)} \mathbf{e}\bigl(-2\lambda\bigl(\xi^\iota \eta\bigr)\bigr) l_\eta$$

for each $\xi \in \Xi$. Hence we obtain

$$\phi_\xi|_k \gamma_p = -p^{-1} \sum_{\eta \in \Xi(p)} \mathbf{e}\bigl(-2\lambda\bigl(\eta^\iota \xi\bigr)\bigr)\phi_\eta.$$

In particular, we get

$$\mathrm{Tr}_p^{D_H}\bigl(\sigma(\phi)\bigr) = \phi_0 + \sum_{j=1}^{p} \phi_0|_k \gamma_p \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \phi_0 - p^{-1} \sum_{j=1}^{p} \sum_{\xi \in \Xi(p)} \mathbf{e}\bigl(-\nu(\xi) j\bigr)\phi_\xi = 0$$

for each prime factor p of D_H, which shows that $\sigma(\phi) \in M_k^+(D_H)$. Moreover, if $\phi \in J_\kappa^M$, then

$$\phi_i = \phi_0 | \prod_{p|D_H} V_{p,i_p},$$

$$\phi_0 | V_{p,i_p} = -\mathbf{a}_p(i_p)^{-1} \sum_{t \in \mathbb{Z}/p\mathbb{Z}} \mathbf{e}\bigl(p^{-1} i_p t\bigr)\phi_0|_k \gamma_p \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

for $i = (i_p)_p \in \bigoplus_{p|D_H} \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/D_H\mathbb{Z}$, which implies the injectivity of σ.

Now we shall show the surjectivity of σ. Granted $\phi_0 \in M_k^+(D_H)$, we define ϕ_i by the formula above, and then set

$$\phi(\tau, u) = \sum_{\xi \in \Xi} \phi_{\rho(\xi)}(\tau)\vartheta_\xi(\tau, u).$$

It is trivial that $\phi|_{\kappa,1}\gamma = \phi$ for $\gamma \in V(\mathbb{Q}) \cap \mathrm{GL}_4(R)$. It is immediate that $\phi|_{\kappa,1}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \phi$ and $\phi|_{\kappa,1}\varepsilon = \phi$ for $\varepsilon \in R^\times$ by direct calculation. Now it remains to show that $\phi|_{\kappa,1}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \phi$. Clearly, it suffices to show that

$$\phi_i|_k \gamma_p = -p^{-1}\Biggl(\phi_0 + \sum_{j \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\eta \in \rho^{-1}(j)} \mathbf{e}\bigl(-2\lambda(\eta\xi)\bigr)\phi_j\Biggr), \quad \xi \in \rho^{-1}(i), \tag{7.1}$$

for each $i \in \mathbb{Z}/p\mathbb{Z}$.

We write $\mathbb{F}_q$ for a finite field with q elements. We identify $\mathbb{Z}/p\mathbb{Z}$ and $\widetilde{R}_p/R_p$ with $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ respectively. Define the character ψ_p of $\mathbb{F}_p$ by $\psi_p(t) = \mathbf{e}(p^{-1}t)$.

If $i=0$, then

$$\begin{aligned}\sum_{j\in\mathbb{F}_p^\times}\sum_{\eta\in\mathbb{F}_{p^2},\rho(\eta)=j}\psi_p\bigl(-\tau(\eta\xi)\bigr)\phi_j &= (p+1)\sum_{j\in\mathbb{F}_p^\times}\phi_j\\ &= -\sum_{j\in\mathbb{F}_p^\times}\sum_{t\in\mathbb{F}_p}\psi_p(tj)\phi_0|_k\gamma_p\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\\ &= -(p-1)\phi_0|\gamma_p+\sum_{t\in\mathbb{F}_p^\times}\phi_0|_k\gamma_p\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\\ &= -p\phi_0|\gamma_p-\phi_0+\mathrm{Tr}_p^{D_H}(\phi_0)=-p\phi_0|\gamma_p-\phi_0\end{aligned}$$

and hence (7.1) holds. Here we view τ and ρ as the trace and the norm of $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$ respectively.

Suppose that $i\in\mathbb{F}_p^\times$. Since

$$\gamma_p\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\gamma_p=\begin{pmatrix}t^{-1} & -1\\ 0 & t\end{pmatrix}\gamma_p\begin{pmatrix}1 & -t^{-1}\\ 0 & 1\end{pmatrix}\pmod p$$

if $t\in\mathbb{F}_p^\times$, we have

$$\begin{aligned}\phi_i|_k\gamma_p &= -(p+1)^{-1}\left(\phi_0+\sum_{t\in\mathbb{F}_p^\times}\psi_p(it)\phi_0|_k\gamma_p\begin{pmatrix}1 & -t^{-1}\\ 0 & 1\end{pmatrix}\right)\\ &= -(p+1)^{-1}\left\{\phi_0-p^{-1}\sum_{t\in\mathbb{F}_p^\times}\psi_p(it)\left(\phi_0+(p+1)\sum_{j\in\mathbb{F}_p^\times}\psi_p\bigl(jt^{-1}\bigr)\phi_j\right)\right\}\\ &= -p^{-1}\left(\phi_0-\sum_{t,j\in\mathbb{F}_p^\times}\psi_p\bigl(it+jt^{-1}\bigr)\phi_j\right).\end{aligned}$$

Finally, the following identity completes our proof:

$$\sum_{\eta\in\mathbb{F}_{p^2},\,\eta\in\rho^{-1}(j)}\psi_p\bigl(-\tau(\eta\xi)\bigr)=-\sum_{t\in\mathbb{F}_p^\times}\psi_p\bigl(it+jt^{-1}\bigr).$$

This is easy and is left to the reader. □

8. Maass spaces on spin and orthogonal groups

First of all we recall some well-known facts on Clifford algebras. The basic reference is [33]. For the time being, we will take V to be a finite dimensional vector space over a field F of characteristic different from 2, and let $\varphi: V\times V\to F$ be a nondegenerate F-bilinear symmetric form. Put $\varphi[x]=\varphi(x,x)$.

A Clifford algebra of φ is an F-algebra A with an F-linear map $p: V\to A$ satisfying the following properties:

- A has an identity element, which we denote by $\mathbf{1}_A$;
- A as an F-algebra is generated by $p(V)$ and $\mathbf{1}_A$;

- $p(x)^2 = \varphi[x]\mathbf{1}_A$ for every $x \in V$;
- A has dimension 2^ℓ over F, where $\ell = \dim V$.

It is known that such a pair (A, p) is unique up to isomorphism. Moreover, p is injective, and as such, V can be viewed as a subspace of A via p. We denote this algebra A by $A(V)$. The basic equalities are

$$xy + yx = 2\varphi(x, y) \quad (x, y \in V).$$

There is an automorphism $\alpha \mapsto \alpha'$ of $A(V)$ such that $v' = -v$ for every $v \in V$. Similarly, there is an anti-automorphism $\alpha \mapsto \alpha^\rho$ of $A(V)$ such that $v^\rho = v$ for every $v \in V$. Let us put

$$A^+(V) = \{\alpha \in A(V) \mid \alpha' = \alpha\},$$
$$G^+(V) = \{\alpha \in A^+(V)^\times \mid \alpha V \alpha^{-1} = V\}.$$

Put $\mu(\alpha) = \alpha\alpha^\rho$ for $\alpha \in G^+(V)$. The map μ gives a homomorphism of $G^+(V)$ to $F^\times$, and the spin group $\mathrm{Spin}^\varphi(F)$ of φ is defined by

$$\mathrm{Spin}^\varphi(F) = \{\alpha \in G^+(V) \mid \mu(\alpha) = 1\}.$$

For $\alpha \in G^+(V)$ we can define $\theta(\alpha) \in \mathrm{GL}(V)$ by $\theta(\alpha)v = \alpha v \alpha^{-1}$ $(v \in V)$. Then it is well known that θ gives an isomorphism of $G^+(V)/F^\times$ onto the special orthogonal group $\mathrm{SO}^\varphi(F)$ of φ. Recall that the spinor norm $\sigma_F : \mathrm{SO}^\varphi(F) \to F^\times/F^{\times 2}$ is defined as follows. Given $g \in \mathrm{SO}^\varphi(F)$, take an element $\alpha \in G^+(V)$ so that $\theta(\alpha) = g$. Then $\mu(\alpha)$ is a well-defined element of $F^\times/F^{\times 2}$ and is denoted by $\sigma_F(g)$. Clearly,

$$\theta\bigl(\mathrm{Spin}^\varphi(F)\bigr) = \{g \in \mathrm{SO}^\varphi(F) \mid \sigma_F(g) = 1\}.$$

Assume that φ is isotropic and consider a decomposition

$$V = Fe_0 + U + Ff_0, \qquad U = (Fe_0 + Ff_0)^\perp,$$
$$e_0^2 = f_0^2 = 0, \qquad e_0 f_0 + f_0 e_0 = 1.$$

The restriction of φ to U is denoted also by φ. Let us define an F-linear map $\Psi : V \to \mathrm{M}_2(A(U))$ by

$$\Psi(re_0 + u + sf_0) = \begin{pmatrix} u & r \\ s & -u \end{pmatrix} \quad (r, s \in F,\ u \in U).$$

Then we can see that $(\mathrm{M}_2(A(U)), \Psi)$ is another Clifford algebra of φ, whence Ψ extends to an isomorphism of $A(V)$ onto $\mathrm{M}_2(A(U))$.

Next assume that (U, φ) has a decomposition

$$(U, \varphi) = Fe + (X, -\varphi_0) + Ff, \qquad X = (Fe + Ff)^\perp,$$
$$e^2 = f^2 = 0, \qquad ef + fe = 1.$$

Specify $F = \mathbb{Q}$ and $X = \mathbb{Q}^t$ (column vectors). Assume further that φ_0 is positive definite. Put $U_v = U \otimes_{\mathbb{Q}} \mathbb{Q}_v$, $U_{\mathbb{C}} = U \otimes_{\mathbb{Q}} \mathbb{C}$, $\varepsilon = e + f$ and

$$\mathfrak{D} = \{\mathcal{X} \in U_\infty \mid \mathcal{X}^2 > 0,\ \varphi(\mathcal{X}, \varepsilon) > 0\},$$
$$\mathcal{D} = \{\mathcal{Z} = \mathcal{X} + \sqrt{-1}\mathcal{Y} \in U_{\mathbb{C}} \mid \mathcal{X} \in U_\infty,\ \mathcal{Y} \in \mathfrak{D}\}.$$

The group $\mathrm{Spin}^\varphi(\mathbb{R})$ acts transitively on $\mathcal{D}$ by $\alpha\mathcal{Z} = (a\mathcal{Z} + b)(c\mathcal{Z} + d)^{-1}$ for $\mathcal{Z} \in \mathcal{D}$ and $\alpha \in \mathrm{Spin}^\varphi(\mathbb{R})$ with $\Psi(\alpha) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Set

$$j(\alpha, \mathcal{Z}) = \mu(c\mathcal{Z} + d), \qquad j_\kappa(\alpha, \mathcal{Z}) = j(\alpha, \mathcal{Z})^\kappa.$$

Put $\mathrm{SO}_0^\varphi(\mathbb{R}) = \{g \in \mathrm{SO}^\varphi(\mathbb{R}) \mid \sigma_{\mathbb{R}}(g) = 1\}$. The action of $\mathrm{SO}_0^\varphi(\mathbb{R})$ on $\mathcal{D}$ and the automorphy factor $j_\kappa(g, \mathcal{Z})$ are defined by

$$g\mathcal{Z}^\sim = (g\mathcal{Z})^\sim j(g, \mathcal{Z}), \qquad j_\kappa(g, \mathcal{Z}) = j(g, \mathcal{Z})^\kappa, \qquad \mathcal{Z}^\sim = \begin{pmatrix} -\varphi[\mathcal{Z}] \\ \mathcal{Z} \\ 1 \end{pmatrix}$$

for $g \in \mathrm{SO}_0^\varphi(\mathbb{R})$. The morphism $\theta : \mathrm{Spin}^\varphi(\mathbb{R}) \to \mathrm{SO}_0^\varphi(\mathbb{R})$ respects the respective actions of the two groups on $\mathcal{D}$. That is

$$\alpha\mathcal{Z} = \theta(\alpha)\mathcal{Z}, \qquad j_\kappa(\alpha, \mathcal{Z}) = j_\kappa\big(\theta(\alpha), \mathcal{Z}\big).$$

Let γ be an element of $\mathrm{Spin}^\varphi(\mathbb{R})$ or of $\mathrm{SO}_0^\varphi(\mathbb{R})$. For any function $G : \mathcal{D} \to \mathbb{C}$ we define $G|_\kappa\gamma : \mathcal{D} \to \mathbb{C}$ by

$$G|_\kappa\gamma(\mathcal{Z}) = j_\kappa(\gamma, \mathcal{Z})^{-1} G(\gamma\mathcal{Z}).$$

Assume that $L_0 = \mathbb{Z}^t$ is a maximal integral lattice of X, i.e., it is maximal with respect to the property $\varphi_0[x] \in \mathbb{Z}$ for all $x \in L_0$. Put

$$L_0^* = \{\beta \in X \mid 2\varphi_0(x, \beta) \in \mathbb{Z} \text{ for every } x \in L_0\},$$
$$T = \mathbb{Z}e + L_0^* + \mathbb{Z}f, \qquad T^+ = T \cap \mathfrak{D}, \qquad L^* = \mathbb{Z}e_0 + T + \mathbb{Z}f_0.$$

We now define subgroups E_v^φ of $\mathrm{SO}^\varphi(\mathbb{Q}_v)$ and E_v^+ of $\mathrm{Spin}^\varphi(\mathbb{Q}_v)$ by

$$E_\infty^\varphi = \{\alpha \in \mathrm{SO}_0^\varphi(\mathbb{R}) \mid \alpha(\sqrt{-1}\varepsilon) = \sqrt{-1}\varepsilon\},$$
$$E_p^\varphi = \{\alpha \in \mathrm{SO}^\varphi(\mathbb{Q}_p) \mid \alpha\mathbb{Z}_p^{t+4} = \mathbb{Z}_p^{t+4}\},$$
$$E_v^+ = \{\alpha \in \mathrm{Spin}^\varphi(\mathbb{Q}_v) \mid \theta(\alpha) \in E_v^\varphi\}.$$

Set $E^\varphi = \prod_v E_v^\varphi$ and $E^+ = \prod_v E_v^+$. Put

$$\Gamma^\varphi = \{\gamma \in \mathrm{SO}_0^\varphi(\mathbb{R}) \mid \gamma\mathbb{Z}^{t+4} = \mathbb{Z}^{t+4}\},$$
$$\Gamma^+ = \{\gamma \in \mathrm{Spin}^\varphi(\mathbb{Q}) \mid \theta(\gamma) \in \Gamma^\varphi\}.$$

We define a subgroup Γ^* of Γ^φ by

$$\Gamma^* = \{\gamma \in \Gamma^\varphi \mid (\gamma - \mathbf{1}_{t+4})L^* \subset \mathbb{Z}^{t+4}\}.$$

Let Q^{φ} be the parabolic subgroup of SO^{φ} which stabilizes the isotropic line spanned by e_0. Put $Q^{+} = \{\alpha \in \mathrm{Spin}^{\varphi} \mid \theta(\alpha) \in Q^{\varphi}\}$. Note that

$$\begin{aligned} \mathrm{Spin}^{\varphi}(\mathbb{A}) &= Q^{+}(\mathbb{A})E^{+}, & \mathrm{Spin}^{\varphi}(\mathbb{Q}) &= Q^{+}(\mathbb{Q})\Gamma^{+}, \\ \mathrm{SO}^{\varphi}(\mathbb{A}) &= Q^{\varphi}(\mathbb{A})E^{\varphi}, & \mathrm{SO}^{\varphi}(\mathbb{Q}) &= Q^{\varphi}(\mathbb{Q})\Gamma^{*}. \end{aligned} \tag{8.1}$$

Definition 8.1. A holomorphic function G on $\mathcal{D}$ is called a cusp form of weight κ with respect to Γ^{+} (resp. Γ^{φ}, Γ^{*}) if $G|_{\kappa}\gamma = G$ for every $\gamma \in \Gamma^{+}$ (resp. Γ^{φ}, Γ^{*}) and has a Fourier expansion of the form

$$G(\mathcal{Z}) = \sum_{\eta \in T^{+}} A_G(\eta)\mathbf{e}\bigl(2\varphi(\eta, \mathcal{Z})\bigr).$$

Let $S_{\kappa}(\Gamma^{+})$ (resp. $S_{\kappa}(\Gamma^{\varphi})$, $S_{\kappa}(\Gamma^{*})$) denote the space of cusp forms of weight κ with respect to Γ^{+} (resp. Γ^{φ}, Γ^{*}).

Definition 8.2. Let $S^{M}_{\kappa}(\Gamma^{\varphi})$ be the subspace of $S_{\kappa}(\Gamma^{\varphi})$ which consists of all functions F having a Fourier expansion of the form

$$G(\mathcal{Z}) = \sum_{\eta \in T^{+}} \sum_{a \in \mathbb{N},\, a \mid \epsilon(\eta)} a^{\kappa-1} c\bigl(a^{-2}\varphi[\eta]\bigr)\mathbf{e}\bigl(2\varphi(\eta, \mathcal{Z})\bigr)$$

for some function c. Here $\epsilon(\eta) = \max\{d \in \mathbb{N} \mid d^{-1}\eta \in T\}$.

Put

$$\Delta^{\varphi} = \left\{ \begin{pmatrix} * & * & * \\ 0 & \mathbf{1}_t & * \\ \mathbf{0}_2 & 0 & * \end{pmatrix} \in \Gamma^{\varphi} \right\}.$$

For a holomorphic function ϕ on $\mathfrak{H}_1 \times X \otimes_{\mathbb{Q}} \mathbb{C}$ we define a function $\widetilde{\phi}$ on $\mathcal{D}$ by

$$\widetilde{\phi}(ze + u + \tau f) = \mathbf{e}(z)\phi(\tau, u) \quad (z, \tau \in \mathfrak{H}_1,\ u \in X \otimes_{\mathbb{Q}} \mathbb{C}).$$

Let $J^{\mathrm{cusp}}_{\kappa}$ be a space of holomorphic functions ϕ on $\mathfrak{H}_1 \times X \otimes_{\mathbb{Q}} \mathbb{C}$ such that

- $\widetilde{\phi}|_{\kappa}\gamma = \widetilde{\phi}$ for every $\gamma \in \Delta^{\varphi}$;
- ϕ possesses a Fourier expansion of the form

$$\phi(\tau, u) = \sum_{r \in L_0^{*},\, N \in \mathbb{N},\, \varphi_0(r) < N} c(N, r)q^{N}\mathbf{e}\bigl(2\varphi_0(r, u)\bigr).$$

The space $J^{\mathrm{cusp}, M}_{\kappa}$ consists of functions $\phi \in J^{\mathrm{cusp}}_{\kappa}$ for which $c(N, r)$ depends only on $N - \varphi_0(r)$.

The following result has been proven by Sugano among others.

Theorem 8.1. *(Cf. [34, Corollary 6.7].) The $\mathbb{C}$-linear map*

$$\begin{aligned} I: & \sum_{r\in L_0^*,\, N\in\mathbb{N},\, \varphi_0(r)<N} c(N,r)q^N \mathbf{e}\big(2\varphi_0(r,u)\big) \\ & \mapsto \sum_{\eta=\begin{pmatrix} a\\ r\\ b\end{pmatrix}\in T^+} \sum_{d\in\mathbb{N},\, d\mid\epsilon(\eta)} d^{\kappa-1}c\big(abd^{-2}, rd^{-1}\big)\mathbf{e}\big(2\varphi(\eta,\mathcal{Z})\big) \end{aligned}$$

yields an injective linear map from J_κ^{cusp} to $S_\kappa(\Gamma^*)$.

Corollary 8.2. *The restriction of I to $J_\kappa^{\mathrm{cusp},M}$ gives a $\mathbb{C}$-linear isomorphism of $J_\kappa^{\mathrm{cusp},M}$ onto $S_\kappa^M(\Gamma^\varphi)$.*

Proof. We write SO_U^φ for the special orthogonal group of (U,φ). Assume that $\phi\in J_\kappa^{\mathrm{cusp},M}$. Then $I(\phi)|_\kappa\gamma = I(\phi)$ for $\gamma\in \mathrm{SO}_U^\varphi(\mathbb{Q})\cap\Gamma^\varphi$. Since Remark 6.5 of [34] asserts that $\mathrm{SO}_U^\varphi(\mathbb{Q})\cap\Gamma^\varphi$ and Γ^* generate Γ^φ, we have $I(\phi)\in S_\kappa(\Gamma^\varphi)$. The remaining parts are easy. $\square$

In what follows we assume that $(X,-\varphi_0)$ is given by $X=H$ with a quaternion algebra H over F and φ_0 is a norm form on H, where F for the moment is an arbitrary field of characteristic different from 2.

Lemma 8.3. *Notation and assumption being as above, we can take*

$$A(U)=\mathrm{M}_4(H), \qquad A^+(U)=\big\{\iota_0(a,b)\ \big|\ a,b\in\mathrm{M}_2(H)\big\}.$$

Moreover, put

$$\Delta=\mathrm{diag}[1,-1,1,-1], \qquad D=\begin{pmatrix} 0 & \delta\\ \delta & 0\end{pmatrix}, \qquad \delta=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

For $\alpha\in\mathrm{M}_4(H)$ we have

$$\alpha'=\Delta\alpha\Delta, \qquad \alpha^\rho = D\alpha^*D,$$

where α^* is the conjugate transpose of α as a matrix over H. Furthermore,

$$G^+(U)=\big\{\iota_0\big(a, t\delta\big(a^{-1}\big)^*\delta\big)\ \big|\ a\in\mathrm{GL}_2(H),\ t\in F^\times,\ \nu(a)=t^2\big\}.$$

Proof. Define a map $p: U\to\mathrm{M}_4(H)$ by

$$p(re+x+sf)=\left(\begin{array}{cc|cc} & -x & & s\\ x^\iota & & s & \\ \hline & r & & -x^\iota\\ r & & x & \end{array}\right) \quad (r,s\in F,\ x\in H).$$

Notice that $p(u)^2=\varphi(u)$ for every $u\in U$. It can easily be verified that elements $p(x)$ for all $x\in V$ generate $\mathrm{M}_4(H)$. Since the other conditions are clearly satisfied, the pair $(\mathrm{M}_4(H),p)$ is a Clifford algebra of (U,φ).

The second assertion holds for every monomial of elements of U, and hence is valid as stated. It follows that

$$A^+(U) = \iota_0\big(\mathrm{M}_2(H), \mathrm{M}_2(H)\big), \qquad \iota_0(a,b)^\rho = \iota_0\big(\delta b^*\delta, \delta a^*\delta\big)$$

for $\iota_0(a,b) \in A^+(U)$. Theorem 7.7 of [32] states that

$$\begin{aligned} G^+(U) &= \big\{\alpha = \iota_0(a,b) \in A^+(U)^\times \,\big|\, \alpha\alpha^\rho \in F^\times,\ \nu(a) = \nu(b) = \big(\alpha\alpha^\rho\big)^2\big\} \\ &= \big\{\iota_0\big(a, t\delta\big(a^{-1}\big)^*\delta\big) \,\big|\, a \in \mathrm{GL}_2(H),\ t \in F^\times,\ \nu(a) = t^2\big\} \end{aligned}$$

as claimed. $\square$

By Lemma 8.3, $A^+(U)$ is isomorphic to the product $\mathrm{M}_2(H) \times \mathrm{M}_2(H)$. We write this isomorphism as $\omega(a) = (\omega_1(a), \omega_2(a))$ with two F-linear ring homomorphisms $\omega_1, \omega_2 : A^+(U) \to \mathrm{M}_2(H)$.

There is another useful isomorphism $\Xi : A(V) \to \mathrm{M}_2(A(U))$ defined by

$$\Xi(\alpha) = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \Psi(\alpha) \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that $\varepsilon = \varepsilon^{-1}$. It should be remarked that $\Xi(A^+(V)) = \mathrm{M}_2(A^+(U))$. Next we define an F-linear ring homomorphism $\Theta : A^+(V) \to \mathrm{M}_4(H)$ by

$$\Theta(\alpha) = \begin{pmatrix} \omega_1(a) & \omega_1(b) \\ \omega_1(c) & \omega_1(d) \end{pmatrix}$$

when $\Xi(\alpha) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For $a \in A(U)$ and $\xi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(A(U))$, we set

$$a^\sigma = \varepsilon a^\rho \varepsilon, \qquad \xi^\sigma = \begin{pmatrix} a^\sigma & c^\sigma \\ b^\sigma & d^\sigma \end{pmatrix}.$$

We obtain

$$\Xi\big(\alpha^\rho\big) = J^{-1}\Xi\big(\alpha'\big)^\sigma J, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for all $\alpha \in A(V)$ as this is so for all $v \in V$. Since $\omega_1(a^\sigma) = \omega_1(a)^*$ for $a \in A^+(U)$, it follows that

$$\Theta(\alpha) \begin{pmatrix} 0 & -\mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \Theta(\alpha)^* = \mu(\alpha) \begin{pmatrix} 0 & -\mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}$$

for $\alpha \in G^+(V)$. Thus Θ restricts to a homomorphism of $\mathrm{Spin}^\varphi(F)$ into $G_2(F)$.

We in turn introduce a homomorphism $x_F : G_n(F) \to F^\times/F^{\times 2}$ analogous to σ_F. For $0 \leqslant j \leqslant n$ let τ_j be a Weyl element given by

$$\tau_j = \left(\begin{array}{cc|cc} & & -\mathbf{1}_j & \\ & \mathbf{1}_{n-j} & & \\ \hline \mathbf{1}_j & & & \\ & & & \mathbf{1}_{n-j} \end{array}\right).$$

Recall that

$$G_n(F) = \bigsqcup_{j=0}^{n} P_n(F)\tau_j P_n(F)$$

is the relative Bruhat decomposition of $G_n(F)$ with respect to $P_n(F)$. If $g = m(a_1)n(b_1)\tau_j m(a_2)n(b_2) \in P_n(F)\tau_j P_n(F)$, then we let $x_F(g) = \nu(a_1a_2)$. Corollary 1.5 of [25] asserts that the coset $x_F(g)F^{\times 2}$ in $F^\times/F^{\times 2}$ is determined by g. Moreover, the map $g \mapsto x_F(g)$ is a homomorphism of $G_n(F)$ into $F^\times/F^{\times 2}$ by Proposition 1.6 of [25] (see also [37]). It should be remarked that there is a misprint in [25]. The factor $(-\epsilon)^t$ should be removed in the relation on p. 372, line 12 in case 2. Granted the homomorphism x_F, we can describe the image of $\mathrm{Spin}^\varphi(F)$ under the morphism Θ as follows:

Lemma 8.4. *In the setting of Lemma* 8.3, *we have*

$$\Theta\big(\mathrm{Spin}^\varphi(F)\big) = \big\{g \in G_2(F) \bigm| x_F(g) = 1\big\}.$$

Remark 8.3. More precisely, the morphism Θ gives rise to an isomorphism

$$\mathrm{Spin}^\varphi(F)/\big\{\omega^{-1}(1,\epsilon) \bigm| \epsilon \in \{\pm 1\}\big\} \simeq \big\{g \in G_2(F) \bigm| x_F(g) = 1\big\}$$

(cf. [9]). We here regard $A^+(U)$ as a subring of $A^+(V)$ in an obvious way. Notice that the center of $\mathrm{Spin}^\varphi(F)$ is $\{\omega^{-1}(\epsilon_1, \epsilon_2) \mid \epsilon_i \in \{\pm 1\}\}$.

Proof. Since $Q^+(F)$ and $(e_0 + f_0)\varepsilon$ generate $\mathrm{Spin}^\varphi(F)$, it suffices to show that $\Theta(Q^+(F))$ and $\Theta((e_0 + f_0)\varepsilon)$ generate the kernel of x_F.

To see this, we recall that

$$\Psi\big(Q^+(F)\big) = \left\{\begin{pmatrix} a & u(a^{-1})^\rho \\ 0 & (a^{-1})^\rho \end{pmatrix} \biggm| a \in G^+(U),\ u \in U\right\}$$

(see Proposition 4.3 of [33]). Lemma 8.3 therefore shows that

$$\Theta\big(Q^+(F)\big) = \big\{m(a)n(b) \bigm| a \in \mathrm{GL}_2(H),\ \nu(a) \in F^{\times 2},\ b \in S_2(H)\big\}.$$

A simple calculation gives $\Theta((e_0 + f_0)\varepsilon) = \tau_2$. The desired fact is now obvious. $\square$

Finally, let $F = \mathbb{Q}$ and assume that H is definite. Now the assignment

$$\iota : re + u + sf \mapsto \begin{pmatrix} r & u \\ u^\iota & s \end{pmatrix}$$

from U to $S_2(H)$ extends to a biholomorphic isomorphism from $\mathcal{D}$ onto $\mathfrak{H}_2$. Taking $\iota(\mathcal{Z}) = \omega_1(\varepsilon\mathcal{Z})$ into account, we can easily verify that the action of $\mathrm{Spin}^\varphi(\mathbb{R})$ on $\mathcal{D}$ can be translated to that of $G_2(\mathbb{R})$ on $\mathfrak{H}_2$ through ι by

$$\iota(\alpha\mathcal{Z}) = \Theta(\alpha)\iota(\mathcal{Z}), \qquad j_\kappa(\alpha, \mathcal{Z}) = j_\kappa\big(\Theta(\alpha), \iota(\mathcal{Z})\big) \tag{8.2}$$

for $\alpha \in \mathrm{Spin}^\varphi(\mathbb{R})$ and $\mathcal{Z} \in \mathcal{D}$.

Proof of Proposition 7.1. Let $\phi \in J_\kappa^{\mathrm{cusp},M}$. Put $G = \jmath(\phi)$. Our task is to show that $G \in S_\kappa(\Gamma_2)$. To see this, we define a function $\widetilde{G} : G_2(\mathbb{A}) \to \mathbb{C}$ via $\widetilde{G}(g) = G(x(\mathbf{i}))j_\kappa(x, \mathbf{i})^{-1}$, writing $g = \rho x w \in P_2(\mathbb{Q})G_2(\mathbb{R})\prod_p C_{2,p}$ as in the proof of Theorem 4.2.

The function G is related to $I(\phi)$ by $G \circ \iota = I(\phi)$, and so by Corollary 8.2, $G \circ \iota$ is a cusp form of weight κ with respect to Γ^φ. We have $G \circ \iota \in S_\kappa(\Gamma^+)$ in view of $\theta(\Gamma^+) \subset \Gamma^\varphi$. By Lemma 8.4, $\Theta(\mathrm{Spin}^\varphi(\mathbb{Q}))$ is a normal subgroup of $G_2(\mathbb{Q})$. Combining this fact with (8.1), for given $\beta \in \mathrm{Spin}^\varphi(\mathbb{Q})$, we can take $\rho' \in Q^+(\mathbb{Q})$ and $\gamma \in \Gamma^+$ so that $\rho^{-1}\Theta(\beta)\rho = \Theta(\rho'\gamma)$. Then we have

$$\begin{aligned}
\widetilde{G}\big(\Theta(\beta)g\big) &= \widetilde{G}\big(\rho\Theta\big(\rho'\gamma\big)x\big) \\
&= \widetilde{G}\big(\Theta(\gamma)x\big) \\
&= G\big(\Theta(\gamma)x(\mathbf{i})\big)j_\kappa\big(\Theta(\gamma), x(\mathbf{i})\big)^{-1} j_\kappa(x, \mathbf{i})^{-1} \\
&= G \circ \iota|_\kappa \gamma\big(\iota^{-1}\big(x(\mathbf{i})\big)\big)j_\kappa(x, \mathbf{i})^{-1} \\
&= G\big(x(\mathbf{i})\big)j_\kappa(x, \mathbf{i})^{-1} = \widetilde{G}(g)
\end{aligned}$$

by virtue of (8.2). Since $\Theta(\mathrm{Spin}^\varphi(\mathbb{Q}))$ and $P_2(\mathbb{Q})$ generate $G_2(\mathbb{Q})$, we conclude that $\widetilde{G} \in \mathfrak{S}_\kappa^2$, thereby completing the proof. $\square$

One bonus of our discussion is an explicit formula for the Siegel series of degree two. This is obtained by employing the homomorphisms θ and Θ.

Proposition 8.5. *Let $h \in T_{2,p} \cap \mathrm{GL}_2(H_p)$. Then we have*

$$\widetilde{F}_{p,h} = \begin{cases} \sum_{a=0}^{\epsilon_p(h)} p^{2a} l_{\mathrm{ord}_p D_h - 2a} & \text{if } p \nmid D_H, \\ \sum_{a=0}^{\epsilon_p(h)} p^{2a}(l_{\mathrm{ord}_p D_h - 2a} - p l_{\mathrm{ord}_p D_h - 2a - 2}) & \text{if } p | D_H. \end{cases}$$

Here $\epsilon_p(h)$ and l_e are defined in Section 2 *and Section* 3 *respectively.*

Remark 8.4.

(1) Under the assumptions (I) and (II) in the introduction, this formula is compatible with the explicit formula for the Fourier coefficients of the Eisenstein series of degree two given in [24, Theorem 3].
(2) Alternatively, there is a direct proof using the formula (9.1) below.

Proof. Writing any $\alpha \in \mathrm{Spin}^\varphi(\mathbb{Q}_p)$ and $g \in \mathrm{SO}^\varphi(\mathbb{Q}_p)$ as

$$\alpha = \wp\varkappa \in Q^+(\mathbb{Q}_p)E_p^+, \qquad g = qw \in Q^\varphi(\mathbb{Q}_p)E_p^\varphi,$$

we can define functions $\varepsilon_{s,p}^+ : \mathrm{Spin}^\varphi(\mathbb{Q}_p) \to \mathbb{C}$ and $\varepsilon_{s,p}^\varphi : \mathrm{SO}^\varphi(\mathbb{Q}_p) \to \mathbb{C}$ by

$$\varepsilon^{+}_{s,p}(\alpha)=\left|\mu(a)\right|_p^s, \qquad \varepsilon^{\varphi}_{s,p}(g)=|t|_p^s,$$

where $a \in G^{+}(U)$ and $t \in \mathbb{Q}_p^{\times}$ are defined by

$$\Psi(\wp)=\begin{pmatrix} a & b(a^{-1})^{\rho} \\ 0 & (a^{-1})^{\rho}\end{pmatrix}, \qquad qe_0=te_0.$$

It immediately follows that

$$\varepsilon_{s,p}\circ\Theta=\varepsilon^{+}_{s,p}=\varepsilon^{\varphi}_{s,p}\circ\theta.$$

For $\eta \in T\otimes_{\mathbb{Z}}\mathbb{Z}_p$ with $\varphi[\eta]\neq 0$ we obtain

$$\begin{aligned} b_p\bigl(\iota(\eta),s\bigr) &= \int\limits_{S_2(H_p)} \varepsilon_{s,p}\left(\begin{pmatrix} \mathbf{1}_2 & 0 \\ x & \mathbf{1}_2\end{pmatrix}\right)\mathbf{e}_p\bigl(-\lambda\bigl(\iota(\eta)x\bigr)\bigr)\,d\mu_p(x) \\ &= \int\limits_{U_p} \varepsilon^{+}_{s,p}\left(\Psi^{-1}\left(\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}\right)\right)\mathbf{e}_p\bigl(-2\varphi(\eta,u)\bigr)\,du \\ &= \int\limits_{U_p} \varepsilon^{\varphi}_{s,p}\left(\theta\circ\Psi^{-1}\left(\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}\right)\right)\mathbf{e}_p\bigl(-2\varphi(\eta,u)\bigr)\,du \end{aligned}$$

by (3.1). The last integral was explicitly calculated by Hirai [11, Theorem 2.1], and our expected formula is an application of his work. □

We call $h \in T_n^{+}$ (or T^{+}) primitive if $\epsilon(h)=1$. The following result, which should be of independent interest, has its origin in Zagier's classical article [38].

Proposition 8.6. *Let*

$$G(\mathcal{Z})=\sum_{\eta\in T^{+}} A_G(\eta)\mathbf{e}\bigl(2\varphi(\eta,\mathcal{Z})\bigr)\in S_\kappa\bigl(\Gamma^{\varphi}\bigr).$$

Then the following conditions are equivalent:

(i) $G \in S_\kappa^M(\Gamma^{\varphi})$;
(ii) $A_G(\eta)=A_G(\eta')$ *whenever* $\epsilon(\eta)=\epsilon(\eta')=1$ *and* $D_\eta=D_{\eta'}$.

Proof. It suffices to prove that (ii) implies (i). Put

$$\phi(\tau,u)=\sum_{r\in L_0^*,\,N\in\mathbb{N}} A_G(e+r+Nf)q^N\mathbf{e}\bigl(2\varphi_0(r,u)\bigr).$$

Then (ii) implies that $\phi \in J_\kappa^{\mathrm{cusp},M}$. Consider a function $G-I(\phi)$, which belongs to $S_\kappa(\Gamma^{\varphi})$ by Corollary 8.2. Since it has vanishing primitive Fourier coefficients by (ii), Theorem 3 of [36] concludes that $G=I(\phi)\in S_\kappa^M(\Gamma^{\varphi})$. □

Corollary 8.7. *Let*

$$F(Z)=\sum_{h\in T_2^+} A_F(h)\mathbf{e}\big(\lambda(hZ)\big)\in S_\kappa(\Gamma_2).$$

Then the following conditions are equivalent:

(a) $F\in S_\kappa^M(\Gamma_2)$;
(b) $A_F(h)=A_F(h')$ *whenever* $\epsilon(h)=\epsilon(h')$ *and* $D_h=D_{h'}$;
(c) $A_F(h)=A_F(h')$ *whenever* $h\approx_p h'$ *for all* p;
(d) $A_F(h)=A_F(h')$ *whenever* $\epsilon(h)=\epsilon(h')=1$ *and* $D_h=D_{h'}$.

Proof. Corollary 2.3 verifies the equivalence of (b) and (c). Using Proposition 7.1 instead of Corollary 8.2, we can adapt the proof above to $F\in S_\kappa(\Gamma_2)$. □

9. Maass spaces in higher degrees

We begin by reviewing a formula for the Siegel series. Fix an element $\sigma\in T_{n,p}\cap \mathrm{GL}_n(H_p)$. Notation being as in Lemma 2.2, we put

$$s_p(\sigma)=\begin{cases} n-\#\{i\mid \sigma_i=J\} & \text{if } p\nmid D_H,\\ n-2\#\{i\mid \sigma_i=\sigma_p(-1)\} & \text{if } p|D_H.\end{cases}$$

We have the congruence $s_p(\sigma)\equiv n \pmod 2$ if p divides D_H. Set

$$\mathcal{D}_p(\sigma)=\mathrm{GL}_n(R_p)\backslash\big\{G\in \mathrm{M}_n(R_p)\cap \mathrm{GL}_n(H_p)\;\big|\;\sigma\big[G^{-1}\big]\in T_{n,p}\big\},$$

$$H_{n,p}(\sigma;X)=\begin{cases} \prod_{j=0}^{s_p(\sigma)-1}(1-p^{2j+1}X) & \text{if } p\nmid D_H,\\ \prod_{j=0}^{s_p(\sigma)/2-1}(1-p^{4j+1}X) & \text{if } p|D_H,\ 2|n,\\ \prod_{j=1}^{(s_p(\sigma)-1)/2}(1-p^{4j-1}X) & \text{if } p|D_H,\ 2\nmid n.\end{cases}$$

Now the formula is

$$\widetilde{F}_{p,\sigma}(X)=\sum_{G\in\mathcal{D}_p(\sigma)} X^{-\operatorname{ord}_p D_\sigma+2\operatorname{ord}_p \nu(G)} H_{n,p}\big(\sigma\big[G^{-1}\big];X^2\big). \tag{9.1}$$

This is a special case of the formula given by Feit [7,8] for local series in a rather general context.

Let $h\in T_n^+$. In analogy with the Siegel modular case [20], we define the map $\rho_h:\mathbb{N}\to\mathbb{C}$ via the relation

$$\sum_{a\in\mathbb{N}}\rho_h(a)a^{-s}=\prod_{p|D_h}\big(1-p^{-s}\big)H_{n,p}\big(h;p^{-s}\big).$$

For each natural number a such that a^2 divides D_h, we set

$$\phi(a;h)=\sum_{G\in\mathcal{D}(h),\ \nu(G)|a}\rho_{h[G^{-1}]}\big(\nu(G)^{-1}a\big),$$

where

$$\mathcal{D}(h) = \mathrm{GL}_n(R)\backslash\bigl\{G \in \mathrm{M}_n(R) \cap \mathrm{GL}_n(H) \bigm| h\bigl[G^{-1}\bigr] \in T_n\bigr\}.$$

It goes without saying that we can naturally identify $\mathcal{D}(h)$ with $\prod_p \mathcal{D}_p(h)$.

For each $N \in \mathbb{N}$ we put $\ell_N(\mathbb{X}) = \prod_p l_{\mathrm{ord}_p N}(X_p) \in \mathcal{R}$. Arguing as in the proof of [20, Theorem 1], we can prove the following equality

$$D_h^{(k-1)/2} \prod_p \widetilde{F}_{p,h}(X_p) = \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi(a;h)\bigl(a^{-2}D_h\bigr)^{(k-1)/2}\ell_{a^{-2}D_h}(\mathbb{X}).$$

Let

$$f(\tau) = \sum_{N\in\mathbb{N}} c_f(N)q^N \in S_k^{\mathrm{new}}(d)$$

be a primitive form as before. Note that

$$c_f(N) = N^{(k-1)/2} \prod_{p|d} (\alpha_p)^{\mathrm{ord}_p N} \prod_{p\nmid d} l_{\mathrm{ord}_p N}(\alpha_p).$$

In particular, if $d = 1$, then

$$A(h) = \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi(a;h)c_f\bigl(a^{-2}D_h\bigr).$$

Thus if we define a $\mathbb{C}$-linear map J_n on formal power series by

$$J_n : \sum_{N\in\mathbb{N}} c(N)q^N \mapsto \sum_{h\in T_n^+} \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi(a;h)c\bigl(a^{-2}D_h\bigr)\mathbf{e}\bigl(\lambda(hZ)\bigr),$$

then J_n gives a linear map from $S_k(\mathrm{SL}_2(\mathbb{Z}))$ to $S_{k+2n-2}(\Gamma_n)$.

Definition 9.1. The space $S_\kappa^M(\Gamma_n)$ is defined as follows: $F \in S_{k+2n-2}(\Gamma_n)$ is an element of $S_{k+2n-2}^M(\Gamma_n)$ if there exists a function $c : \mathbb{N} \to \mathbb{C}$ such that all $h \in T_n^+$ satisfy

$$A_F(h) = \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi(a;h)c\bigl(a^{-2}D_h\bigr).$$

As one can easily check by means of Proposition 8.5, Definition 9.1 is consistent with Definition 4.1 when $n = 2$ (see the proof of Theorem 4.3 below).

For each $N \in \mathbb{N}$ we define $\ell'_N \in \mathcal{R}$ by

$$\ell'_N(\mathbb{X}) = \mathbf{b}(N)^{-1} \prod_{p|D_H} \bigl(l_{\mathrm{ord}_p N}(X_p) - p l_{\mathrm{ord}_p N-2}(X_p)\bigr) \prod_{p\nmid D_H} l_{\mathrm{ord}_p N}(X_p),$$

where we put $\mathbf{b}(N)=\prod_{p|D_H}\mathbf{b}_p(N)$ with

$$\mathbf{b}_p(N)=\begin{cases}1 & \text{if } p\nmid N,\\ p+1 & \text{if } p|N.\end{cases}$$

Choosing complex numbers $\phi'(a;h)$ in such a way that

$$D_h^{(k-1)/2}\prod_p \widetilde{F}_{p,h}(X_p)=\sum_{a\in\mathbb{N},\,a^2|D_h} a^{k-1}\phi'(a;h)\big(a^{-2}D_h\big)^{(k-1)/2}\ell'_{a^{-2}D_h}(\mathbb{X}),$$

we define another $\mathbb{C}$-linear map J'_n on formal power series by

$$J'_n:\sum_{N\in\mathbb{N}}c(N)q^N\mapsto\sum_{h\in T_n^+}\sum_{a\in\mathbb{N},\,a^2|D_h}a^{k-1}\phi'(a;h)c\big(a^{-2}D_h\big)\mathbf{e}\big(\lambda(hZ)\big).$$

Put $\omega_{D_H}=\left(\begin{smallmatrix}0 & -1\\ D_H & 0\end{smallmatrix}\right)$. Recall that there is a map from $S_k^{\mathrm{new}}(d)$ to $S_k^+(D_H)$, which we have written $f\mapsto f^*$ (see Lemma 7.2).

Lemma 9.1. *Notation being as above, put $g=f^*|_k\omega_{D_H}$. Then $J'_n(g)$ is equal to* $\mathrm{Lift}_n(f)$ *up to scalar multiple.*

Proof. Since $\alpha_p^2=p^{-1}$ for each prime factor p of d, we have

$$N^{(k-1)/2}\ell'_N\big(\{\alpha_p\}\big)=\sum_{a\in\mathbb{N},\,a|d^{-1}D_H}\mathbf{b}_{d^{-1}D_H}(N)^{-1}\mu(a)a^k c_f\big(a^{-2}N\big).$$

Here, μ denotes the Möbius function and $\mathbf{b}_{d^{-1}D_H}(N)=\prod_{p|d^{-1}D_H}\mathbf{b}_p(N)$.

On the other hand, for each prime factor p of $d^{-1}D_H$, we have

$$\begin{aligned}f|Q(p)W(p)&=f|T(p)\begin{pmatrix}p & \\ & 1\end{pmatrix}-(p+1)p^{k/2-1}f\\ &=-p^{k/2-1}(p+1)\sum_{N\in\mathbb{N}}\mathbf{b}_p(N)^{-1}\big(c_f(N)-p^kc_f\big(p^{-2}N\big)\big)q^N,\end{aligned}$$

where $W(p)$ denotes the Atkin–Lehner involution. Recall that $f|W(p)=-\epsilon_pf$ if d is divisible by p. It follows that g is equal to $\sum_{N\in\mathbb{N}}N^{(k-1)/2}\ell'_N(\{\alpha_p\})q^N$ up to scalar multiple. Thus $J'_n(g)$ is equal to

$$\begin{aligned}&\sum_{h\in T_n^+}\sum_{a\in\mathbb{N},\,a^2|D_h}a^{k-1}\phi'(a;h)\big(a^{-2}D_h\big)^{(k-1)/2}\ell'_{a^{-2}D_h}\big(\{\alpha_p\}\big)\mathbf{e}\big(\lambda(hZ)\big)\\ &\quad=\sum_{h\in T_n^+}D_h^{(k-1)/2}\prod_p\widetilde{F}_{p,h}(\alpha_p)\mathbf{e}\big(\lambda(hZ)\big)\end{aligned}$$

up to scalar multiple. □

Proof of Theorem 4.3. When $n=2$, Proposition 8.5 shows that

$$\phi'(a;h)=\begin{cases}a^2\mathbf{b}(a^{-2}D_h) & \text{if } a|\epsilon(h),\\ 0 & \text{if } a\nmid\epsilon(h).\end{cases}$$

Let us define a $\mathbb{C}$-linear map ς on the formal power series by

$$\varsigma : \sum_{N\in\mathbb{N}} c(N)q^N \mapsto \sum_{\xi\in\widetilde{R},\, N\in\mathbb{N}} \mathbf{b}\big(D_H\big(N-\nu(\xi)\big)\big)c\big(D_H\big(N-\nu(\xi)\big)\big)q^N\mathbf{e}\big(2\lambda(\xi u)\big).$$

An examination of the proof of Proposition 7.3 confirms that the map $h \mapsto \varsigma(h|_k\omega_{D_H})$ turns out to define the inverse map of σ (up to scalar multiple). Observe that $J_2' = \jmath \circ \varsigma$. Lemma 9.1 asserts that $\mathrm{Lift}_2(f)$ equals $\jmath(\varsigma(f^*|_k\omega_{D_H}))$ up to scalar multiple. In summary, the following diagram is commutative up to a scalar:

$$\begin{tikzcd}
f \in \bigoplus_{d\geqslant 1,\, d|D_H} S_k^{\mathrm{new}}(d) \arrow[r, "\mathrm{Lift}_2"] \arrow[d, mapsto] & S_{k+2}^M(\Gamma_2) \\
f^*|_k\omega_{D_H} \in S_k^+(D_H)|_k\omega_{D_H} \arrow[r, "\varsigma"'] \arrow[ur, "J_2'"] & J_{k+2}^{M,\mathrm{cusp}} \arrow[u, "\jmath"']
\end{tikzcd}$$

Here, by abuse of notation Lift_2 stands for the $\mathbb{C}$-linear extension of the map $f \mapsto \mathrm{Lift}_2(f)$ defined on primitive forms in Section 4. Now Propositions 7.1 and 7.3 and Lemma 7.2 combine to give Theorem 4.3. □

Proposition 9.2.

(1) *Suppose that $F \in S_{k+2n-2}(\Gamma_n)$ has a Fourier expansion of the form*

$$F(Z) = \sum_{h\in T_n^+} \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi(a;h)c\big(a^{-2}D_h\big)\mathbf{e}\big(\lambda(hZ)\big)$$

for some function $c : \mathbb{N} \to \mathbb{C}$. If n is odd, then

$$\sum_{N\in\mathbb{N}} c(N)q^N \in S_k\big(\mathrm{SL}_2(\mathbb{Z})\big).$$

(2) *Suppose that $F \in S_{k+2n-2}(\Gamma_n)$ has a Fourier expansion of the form*

$$F(Z) = \sum_{h\in T_n^+} \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\phi'(a;h)c\big(a^{-2}D_h\big)\mathbf{e}\big(\lambda(hZ)\big)$$

for some function $c : \mathbb{N} \to \mathbb{C}$. If n is even, then

$$\sum_{N\in\mathbb{N}} c(N)q^N \in S_k^+(D_H)|_k\omega_{D_H}.$$

Proof. Put $t = [\frac{n-1}{2}]$, $m = 2t$ and $r = n - m$. Lemma 2.4 enables us to choose $S \in T_m$ in such a way that $D_S = 1$. Let l_0 be the characteristic function of the set $\prod_p \mathrm{M}_{mr}(R_p)$. Since $\vartheta_{l_0}^S(Z,u) \in J_{2m,S}(\Gamma_{m,r})$ (but if m were odd, this would not be the case), the Sth Fourier–Jacobi coefficient F_S of F is written as

$$F_S(Z,u) = F_{S,0}(Z)\vartheta_{l_0}^S(Z,u),$$

$$F_{S,0}(Z) = \sum_{h\in T_r^+} A_F\left(\begin{pmatrix} S & 0 \\ 0 & h\end{pmatrix}\right)\mathbf{e}\big(\lambda(hZ)\big) \in S_{k+2r-2}(\Gamma_r)$$

as a consequence of the invariance of F_S with respect to $V(\mathbb{Q}) \cap \mathrm{GL}_{2n}(R)$.

Employing (9.1), we can prove

$$\widetilde{F}_{p,\mathrm{diag}[S,h]} = \widetilde{F}_{p,h}$$

by exactly the same argument given for the proof of [21, (6)]. It follows that

$$A_F\left(\begin{pmatrix} S & 0 \\ 0 & h\end{pmatrix}\right) = \sum_{a\in\mathbb{N},\, a^2|D_h} a^{k-1}\varphi(a;h)c\big(a^{-2}D_h\big),$$

where $\varphi(a;h) = \phi(a;h)$ or $\phi'(a;h)$ according as n is odd or even. Thus the proof is reduced to the case $n=1$ or $n=2$. There is nothing to prove if $n=1$, and this is the situation of Theorem 4.3 as just described if $n=2$. $\square$

Corollary 9.3. *If n is odd, then $S^M_{k+2n-2}(\Gamma_n) = J_n(S_k(\mathrm{SL}_2(\mathbb{Z})))$.*

If n is odd and if we replace F with $\mathrm{Lift}_n(f)$ in the proof above, then

$$F_{S,0}(\tau) = \sum_{N\in\mathbb{N}} N^{(k-1)/2}\ell_N\big(\{\alpha_p\}\big)q^N.$$

In case $d>1$, this is definitely not a modular form, and hence neither is $F = \mathrm{Lift}_n(f)$, even though one allows it to have a level.

10. Open problems

There are several naturally arising open problems.

It is natural to ask for the existence of a lifting that generalizes the liftings described in Theorems 4.2 and 4.3. Let F be a totally real number field and H a totally definite quaternion algebra over F. Let $\mathfrak{S}_\infty$ be the set of archimedean places of F and $\mathfrak{S}_H$ the set of finite places of F at which H is ramified. Let $\tau \simeq \bigotimes_v \tau_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$, on which we impose the following conditions:

(i) τ_v is a discrete series with extremal weight $\pm k_v$ for $v\in\mathfrak{S}_\infty$;
(ii) there is a (possibly empty) subset $\mathfrak{S}_d$ of $\mathfrak{S}_H$ such that:
- τ_v is a twisted Steinberg representation $St_v \otimes \chi_v$ for $v\in\mathfrak{S}_d$;
- τ_v is a principal series $\pi(\mu_v,\mu_v^{-1})$ for $v\notin\mathfrak{S}_\infty\cup\mathfrak{S}_d$.

Since τ_v has trivial central character, k_v must be even and $\chi_v^2=1$. We consider the representation $\Pi_n(\tau) = \bigotimes_v \Pi_{n,v}(\tau_v)$ of $G_n(\mathbb{A}_F)$ by setting

$$\Pi_{n,v}(\tau_v) = \begin{cases} \mathcal{D}^n_{(k_v+2n-2)/2} & \text{if } v\in\mathfrak{S}_\infty, \\ A^n_v(1)\otimes(\chi_v\circ x_{F_v}) & \text{if } v\in\mathfrak{S}_d, \\ I_{n,v}(0,\mu_v) & \text{if } v\notin\mathfrak{S}_\infty\cup\mathfrak{S}_d \end{cases}$$

(see Section 8 for the definition of x_{F_v}).

Conjecture 10.1. *Notation being as above, $\Pi_n(\tau)$ is automorphic.*

In classical language we set forth the following:

Conjecture 10.2. *Notation being as in Section 4, we assume that n is even. Then $\mathrm{Lift}_n(f)$ is a cuspidal Hecke eigenform in $S^M_{k+2n-2}(\Gamma_n)$. Moreover, $\Pi_n(f)$ is isomorphic to the cuspidal automorphic representation associated to $\mathrm{Lift}_n(f)$. Furthermore, the lifting $f \mapsto \mathrm{Lift}_n(f)$ gives a bijection (up to a scalar) between Hecke eigenforms in $\bigoplus_{d\geqslant 1, d|D_H} S_k^{\mathrm{new}}(d)$ and those in $S^M_{k+2n-2}(\Gamma_n)$.*

Remark 10.1.

(1) Conjecture 10.1 is true by Theorem 4.2 if τ comes from an elliptic cusp form relative to $\mathrm{SL}_2(\mathbb{Z})$. Conjecture 10.1 is true as well if $n=1$. Indeed, let $\widetilde{f}:\mathrm{GL}_2(\mathbb{A}_F)\to\mathbb{C}$ is a function belonging to τ. The accidental isomorphism (1.1) allows us to define a function $\mathcal{F}: G_1(\mathbb{A}_F)\to\mathbb{C}$ by
$$\mathcal{F}(\alpha\gamma)=\widetilde{f}(\gamma)$$
for $\alpha\in\mathrm{GL}_1(H\otimes_F\mathbb{A}_F)$ and $\gamma\in\mathrm{GL}_2(\mathbb{A}_F)$ subject to $\nu(\alpha)\det\gamma=1$. Then $\mathcal{F}$ is well defined and is a vector of $\Pi_1(\tau)$.

(2) When $n=2$, Conjecture 10.2 is true by Theorem 4.3. Lemma 9.1 and Proposition 9.2 assert that Conjecture 10.2 is equivalent to the following statement: if n is even, then J'_n gives rise to a $\mathbb{C}$-linear map from $S_k^+(D_H)|_k\omega_{D_H}$ to $S_{k+2n-2}(\Gamma_n)$.

(3) If n is odd and $d>1$, then for each prime factor p of d, the value at $s=(\log\alpha_p)/(\log p)$ of the $C_{n,p}$-invariant section $\varepsilon_{s,p}\in I_{n,p}(s)$ does not lie in the space $A^n_p(\epsilon_p)$ by Proposition 4.1(2). Note that $B^n_p(\epsilon_p)$ can never arise as local components of cusp forms as it is singular. This explains the reason why $\mathrm{Lift}_n(f)$ is far from being a modular form in this case. If we assume Conjecture 10.2, then $\Pi_n(f)$ will be generated by the components of the first Fourier–Jacobi coefficient of $\mathrm{Lift}_{n+1}(f)$.

When $n>2$, the definition of the space $S^M_\kappa(\Gamma_n)$ may look technical. Let $G_\kappa(\Gamma_n)$ be as in the introduction. It is evident that $G_\kappa(\Gamma_n)$ includes $S^M_\kappa(\Gamma_n)$.

Conjecture 10.3. *The spaces $S^M_\kappa(\Gamma_n)$ and $G_\kappa(\Gamma_n)$ coincide.*

Remark 10.2.

(1) Corollary 8.7 tells us that Conjecture 10.3 is affirmative if $n=2$.

(2) Kitaoka [19] defined the Siegel modular analogue of the space $G_\kappa(\Gamma_n)$ and showed that it is closed under the action of Hecke operators. One can verify that $G_\kappa(\Gamma_n)$ is also Hecke invariant by substantially the same way.

If the following statement were available, then proofs of Theorems 4.2 and 4.3 would be rather simpler.

Conjecture 10.4. *The group Γ_n is generated by $\Gamma_n\cap P_n(\mathbb{Q})$ and $\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0\end{pmatrix}$.*

Under the assumptions (I) and (II) given in the introduction, Conjecture 10.4 is affirmative by Theorem 2.3 in Chapter 2 of [22].

Appendix A. Automorphic *L*-functions associated to automorphic forms on quaternionic unitary groups

A little more generally, let F be a number field with ring of adeles $\mathbb{A}_F$. Let $\mathfrak{S}_\infty$ be the set of archimedean places of F and $\mathfrak{S}_H$ the set of finite places of F for which H ramifies. Recall first of all that the L-group of G_n is ${}^L G_n = \mathrm{SO}_{4n}(\mathbb{C}) \times W_F$, where W_F is the Weil group of F. Let $\mathrm{st}: {}^L G_n \to \mathrm{GL}_{4n}(\mathbb{C})$ be the standard representation on $\mathbb{C}^{4n}$ on the first factor and trivial on W_F. If $\pi \simeq \bigotimes_v \pi_v$ is an irreducible cuspidal automorphic representation of $G_n(\mathbb{A}_F)$, then for all places v outside of a finite set S which includes $\mathfrak{S}_\infty \cup \mathfrak{S}_H$, the local component π_v is the spherical constituent of an unramified principal series representation. Letting $t_v \in \mathrm{SO}_{4n}(\mathbb{C})$ be its Satake parameter, we define the local Euler factor attached to π_v by

$$L_v(s, \pi_v, \mathrm{st}) = \det\bigl(1 - q_v^{-s} \cdot \mathrm{st}(t_v \times Fr_v)\bigr)^{-1},$$

where Fr_v is a Frobenius element of W_{F_v}. The standard L-function of π is then

$$L^S(s, \pi, \mathrm{st}) = \prod_{v \notin S} L_v(s, \pi_v, \mathrm{st}).$$

Let Φ be a standard section of $I_{2n}(s, \chi)$ and E_Φ the corresponding Eisenstein series on $G_{2n}(\mathbb{A}_F)$. For a function $\mathcal{F} \in \pi$ we consider the doubling integral defined by

$$Z(s, \Phi, \mathcal{F})(x) = \int\limits_{G_n(F)\backslash G_n(\mathbb{A}_F)} E_\Phi\bigl(\iota_0(g, \check{x})\bigr)\mathcal{F}(g)\, dg,$$

where

$$\check{x} = \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix} x \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix}.$$

Note that this is not the standard doubling integral considered by Piatetski-Shapiro and Rallis [28] but of the type considered by Böcherer [4] since we are integrating an Eisenstein series against only one cusp form.

Let

$$\delta = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & 1 & & \\ & & 1 & 1 \end{pmatrix} \in G_{2n}(F),$$

where the blocks have size $n \times n$. Unfolding the Eisenstein series in the usual way, we get

$$\begin{aligned} Z(s, \Phi, \mathcal{F})(x) &= \int\limits_{G_n(\mathbb{A}_F)} \Phi\bigl(\delta\iota_0(g, \mathbf{1}_{2n})\bigr)\mathcal{F}(xg)\, dg \\ &= \prod_v Z_v(s, \Phi_v, \mathcal{F}_v)(x_v), \end{aligned}$$

for $\Re s \gg 0$, where we have assumed that $\Phi = \bigotimes_v \Phi_v$ and $\mathcal{F} = \bigotimes_v \mathcal{F}_v$ are factorisable and written

$$Z_v(s, \Phi_v, \mathcal{F}_v) = \int\limits_{G_n(F_v)} \Phi_v\bigl(\delta\iota_0(g, \mathbf{1}_{2n})\bigr)\pi_v(g)\mathcal{F}_v\, dg$$

in the last step. The integral $\mathcal{F}_v \to Z_v(s, \Phi_v, \mathcal{F}_v)$ here gives rise to an endomorphism of the local component π_v.

In what follows, we take $\chi = 1$ and limit ourselves to the case that π_v contains a nonzero $C_{n,v}$-invariant vector $\mathcal{F}_v^0$. Let Φ_v^0 be the normalized $C_{2n,v}$-invariant vector of $I_{2n,v}(s)$. For $v \notin \mathfrak{S}_\infty \cup \mathfrak{S}_H$, the unramified vector $\mathcal{F}_v^0$ is an eigenvector for the operator $Z_v(s, \Phi_v^0, \cdot\,)$ with eigenvalue

$$L_v\left(s+\frac{1}{2}, \pi_v, \mathrm{st}\right)\prod_{j=1}^{2n}\left(1-q_v^{1-2s-2j}\right)$$

by Proposition 6.2 of [28] (though [28] considers the full orthogonal groups, this does not affect the result).

Next we look at the case that $v \in \mathfrak{S}_H$. Let B_n be a parabolic subgroup of G_n defined by

$$B_n = \left\{m(a)u \in P_n \;\middle|\; a \text{ is upper triangular}\right\}.$$

For $\mu \in \mathbb{C}^n$ the induced representation $\mathrm{Ind}_{B_n(F_v)}^{G_n(F_v)}(\mu)$ is given by the right multiplication action of $G_n(F_v)$ on the space of smooth functions f on $G_n(F_v)$ satisfying

$$f\left(m(a)ug\right) = \left|\nu(a_1)\right|_v^{\mu_1+2n-3/2}\left|\nu(a_2)\right|_v^{\mu_2+2n-7/2}\cdots\left|\nu(a_n)\right|_v^{\mu_n+1/2} f(g),$$

where $a = \mathrm{diag}[a_1, \dots, a_n] \in \mathrm{GL}_n(H_v)$. Assume that π_v is the constituent of $\mathrm{Ind}_{B_n(F_v)}^{G_n(F_v)}(\mu)$ containing the right $C_{n,v}$-invariant function $\mathcal{F}_v^0$ with $\mathcal{F}_v^0(1) = 1$. Since the space of $C_{n,v}$-invariants in $\mathrm{Ind}_{B_n(F_v)}^{G_n(F_v)}(\mu)$ is one dimensional, $\mathcal{F}_v^0$ is unique and hence $Z_v(s, \Phi_v^0, \mathcal{F}_v^0)$ is a multiple of $\mathcal{F}_v^0$. More precisely, we obtain the following formula by the calculation analogous to that of [28].

Proposition A.1. *Notation and assumption being as above, we have*

$$Z_v\left(s, \Phi_v^0, \mathcal{F}_v^0\right)(x) = \mathcal{F}_v^0(x)\prod_{j=1}^{n}\frac{1-q_v^{1-2s-4j}}{(1-q_v^{-s-1-\mu_j})(1-q_v^{-s-1+\mu_j})}.$$

On the other hand, for $v \notin \mathfrak{S}_\infty$

$$\Phi_v^0\left(\delta\iota_0(g, \mathbf{1}_{2n})\right) = \nu[g]^{-(s+2n-1/2)}, \quad g \in G_n(F_v),$$

where $\nu[g] = [gR_v^{2n} + R_v^{2n} : R_v^{2n}]^{1/2}$. This is Proposition 6.4 of [28], provided that $v \notin \mathfrak{S}_H$. As is easily checked, this formula remains valid when $v \in \mathfrak{S}_H$. It follows that

$$Z_v\left(s, \Phi_v^0, \mathcal{F}\right)(x) = \sum_{\beta \in C_{n,v}\backslash G_n(F_v)/C_{n,v}} (\mathcal{F}|C_{n,v}\beta C_{n,v})(x)\nu[\beta]^{-(s+2n-1/2)},$$

where $\mathcal{F}|C_{n,v}\beta C_{n,v} : G_n(\mathbb{A}_F) \to \mathbb{C}$ is given by

$$(\mathcal{F}|C_{n,v}\beta C_{n,v})(x) = \sum_{\eta \in \mathcal{B}_\beta}\mathcal{F}\left(x\eta^{-1}\right),$$

taking a finite subset $\mathcal{B}_\beta$ of $G_n(F_v)$ so that $C_{n,v}\beta C_{n,v} = \bigsqcup_{\eta \in \mathcal{B}_\beta} C_{n,v}\eta$.

Finally, we prove the assertions concerning the local components of $\mathrm{Lift}_n(f)$. We may assume that $n \geqslant 2$ since our assertion is clear when $n = 1$ (see Remarks 4.2(1) and 10.1(1)). Put $\widetilde{E}_\kappa^n(g) = \widetilde{E}_\kappa^n(g, 0)$.

The ground field being $\mathbb{Q}$ and notation being as in the proof of Theorem 4.2, we can show that there exists a polynomial Φ_β satisfying

$$\widetilde{E}^n_\kappa | C_{n,p}\beta C_{n,p} = \Phi_\beta\bigl(p^{(k-1)/2}\bigr)\widetilde{E}^n_\kappa, \qquad \mathcal{F}|C_{n,p}\beta C_{n,p} = \Phi_\beta(\alpha_p)\mathcal{F}$$

for sufficiently large κ by the same way as [14].

Using the computations above, we get

$$\sum_{\beta\in C_{n,p}\backslash G_n(\mathbb{Q}_p)/C_{n,p}} \Phi_\beta\bigl(p^{(k-1)/2}\bigr)\nu[\beta]^{-(s+2n-1/2)}$$
$$= \begin{cases} \prod_{j=1}^{2n} \frac{1-p^{1-2s-2j}}{(1-p^{-s+(k-1)/2-n+j-1})(1-p^{-s-(k-1)/2+n-j})} & \text{if } p\nmid D_H, \\ \prod_{j=1}^{n} \frac{1-p^{1-2s-4j}}{(1-p^{-s+(k-1)/2-n+2j-2})(1-p^{-s-(k-1)/2+n-2j})} & \text{if } p|D_H. \end{cases}$$

From this, we infer that

$$\sum_{\beta\in C_{n,p}\backslash G_n(\mathbb{Q}_p)/C_{n,p}} \Phi_\beta(\alpha_p)\nu[\beta]^{-(s+2n-1/2)}$$
$$= \begin{cases} \prod_{j=1}^{2n} \frac{1-p^{1-2s-2j}}{(1-\alpha_p p^{-(s+1/2)-n+j-1/2})(1-\alpha_p^{-1} p^{-(s+1/2)-n+j-1/2})} & \text{if } p\nmid D_H, \\ \prod_{j=1}^{n} \frac{1-p^{1-2s-4j}}{(1-\alpha_p p^{-(s+1)-n+2j-1})(1-\alpha_p^{-1} p^{-(s+1)-n+2j-1})} & \text{if } p|D_H. \end{cases}$$

Recalling that $\varepsilon_{\kappa,0} \in I_n(\frac{k-1}{2})$, we can easily see that the local components of $\mathrm{Lift}_n(f)$ precisely coincide with those of $\Pi_n(f)$. $\square$

Appendix B. An interpretation in terms of Arthur's conjecture

We explain how Conjecture 10.1 can be viewed in the framework of Arthur's conjecture. For details of the conjecture, the reader should consult [2].

Let $\mathcal{L}_F$ be the hypothetical Langlands group over F. By a (discrete) A-parameter for G_n, we mean an equivalence class of homomorphisms $\psi : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_n^0 = \mathrm{SO}_{4n}(\mathbb{C})$ which satisfy the following conditions:

- the restriction of ψ to $\mathcal{L}_F$ has bounded image;
- the restriction of ψ to $\mathrm{SL}_2(\mathbb{C})$ is an algebraic homomorphism;
- the centralizer of the image of ψ in ${}^L G_n^0$ is finite.

We write Z_ψ for this centralizer and define the global S-group $\mathcal{S}_\psi$ to be the quotient of Z_ψ by the center of ${}^L G_n^0$.

The global A-parameter ψ gives rise to the local A-parameter

$$\psi_v : \mathcal{L}_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_n^0,$$

where $\mathcal{L}_{F_v}$ is the Weil group of F_v if $v \in \mathfrak{S}_\infty$, and the Weil–Deligne group of F_v if $v \notin \mathfrak{S}_\infty$. Arthur's conjecture predicts that there exists a finite set Π_{ψ_v} of equivalence classes of unitary admissible representations of $G_n(F_v)$ associated to ψ_v. When $G_n(F_v)$ is quasisplit, it is imposed that Π_{ψ_v} contains the L-packet associated to the Langlands parameter $\phi_{\psi_v} : \mathcal{L}_{F_v} \to {}^L G_n^0$ given by

$$\phi_{\psi_v}(w) = \psi_v\left(w, \begin{pmatrix} |w|_v^{1/2} & \\ & |w|_v^{-1/2} \end{pmatrix}\right).$$

With the local packet Π_{ψ_v} at hand, we define the global packet by

$$\Pi_\psi = \left\{ \bigotimes_v \pi_v \,\middle|\, \pi_v \in \Pi_{\psi_v} \right\}.$$

We here understand that for almost all v, π_v is the unramified representation whose Satake parameter is $\phi_{\psi_v}(Fr_v)$.

It is believed that there exists a pairing $\langle\ ,\ \rangle : \mathcal{S}_\psi \times \Pi_\psi \to \mathbb{C}$ (Arthur [2] discussed this object locally). Arthur attached to ψ a quadratic character ϵ_ψ of $\mathcal{S}_\psi$. For each $\pi \in \Pi_\psi$, set

$$m_\psi(\pi) = \frac{1}{\#\mathcal{S}_\psi} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s)\langle s, \pi\rangle.$$

Arthur conjectures that the multiplicity of π in the space of square-integrable automorphic forms on $G_n(\mathbb{A})$ is equal to $\sum_{\pi \in \Pi_\psi} m_\psi(\pi)$.

If τ is a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$, then τ corresponds to a map $\rho_\tau : \mathcal{L}_F \to \mathrm{SL}_2(\mathbb{C})$. Let Sym^{2n-1} be the irreducible $2n$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. As is well known, we may assume that $\mathrm{Sym}^{2n-1}(\mathrm{SL}_2(\mathbb{C})) \subset \mathrm{Sp}_n(\mathbb{C})$. Now consider the global A-parameter

$$\psi_\tau = \rho_\tau \boxtimes \mathrm{Sym}^{2n-1} : \mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C}) = {}^L G_n^0.$$

If τ comes from an elliptic cusp form f which is as in Theorem 4.2 or 4.3, then the L-parameter of $\Pi_{n,v}(f)$ agrees with $\phi_{\psi_{\tau,v}}$ for every $v \notin \mathfrak{S}_\infty \cup \mathfrak{S}_H$, and as such, the A-parameter of $\mathrm{Lift}_n(f)$ should be ψ_τ.

The point here is that when $G_n(F_v)$ is not quasisplit, Arthur's conjecture does not specify any representations in $\Pi_{\psi_{\tau,v}}$. If τ is as in Conjecture 10.1, in accordance with Theorems 4.2 and 4.3, we guess that the packet $\Pi_{\psi_{\tau,v}}$ contains $\Pi_{n,v}(\tau_v)$ for every v. Since $\mathcal{S}_{\psi_\tau} = \{1\}$, Arthur's conjectural multiplicity formula says that any element of Π_{ψ_τ} should be automorphic.

References

[1] J. Arthur, Eisenstein series and the trace formula, Proc. Sympos. Pure Math. 33 (1) (1979) 253–274.
[2] J. Arthur, Unipotent automorphic representations: conjectures, Astérisque 171–172 (1989) 13–71.
[3] D. Ban, C. Jantzen, Degenerate principal series for even-orthogonal groups, J. Amer. Math. Soc. 7 (2003) 440–480.
[4] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, J. Reine Angew. Math. 362 (1985) 146–168.
[5] S. Böcherer, W. Kohnen, On the functional equation of singular series, Abh. Math. Sem. Univ. Hamburg 70 (2000) 281–286.
[6] P. Feit, Poles and residues of Eisenstein series for symplectic and unitary groups, Mem. Amer. Math. Soc. 346 (1986).
[7] P. Feit, Explicit formulas for local factors in the Euler products for Eisenstein series, Nagoya Math. J. 113 (1989) 37–87.
[8] P. Feit, Explicit formulas for local factors: addenda and errata, Nagoya Math. J. 133 (1994) 177–187.
[9] E. Freitag, C.F. Hermann, Some modular varieties of low dimension, Adv. Math. 152 (2000) 203–287.
[10] V. Gritsenko, Fourier–Jacobi functions in n variables, Zap. Nauchn. Sem. Leningr. Otdel. Mat. Inst. Steklov. 168 (1988) 32–44; Soviet Math. 53 (1991) 242–252.
[11] Y. Hirai, Eisenstein series on orthogonal groups $O(1, m+1)$ and $O(2, m+2)$, Hiroshima Math. J. 28 (1998) 7–54.
[12] R. Howe, On a notion of rank for unitary representations of classical groups, in: Harmonic Analysis and, Group Representations, Liguori, Naples, 1982, pp. 223–331.
[13] T. Ikeda, On the theory of Jacobi forms and the Fourier–Jacobi coefficients of Eisenstein series, J. Math. Kyoto Univ. 34 (1994) 615–636.

[14] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$, Ann. of Math. 154 (2001) 641–681.
[15] T. Ikeda, On the lifting of hermitian modular forms, Compos. Math. 144 (2008) 1107–1154.
[16] B. Jacobowitz, Hermitian forms over local fields, Amer. J. Math. 84 (1962) 441–465.
[17] H. Katsurada, An explicit formula for Siegel series, Amer. J. Math. 121 (1999) 415–452.
[18] H. Kim, Eisenstein series on quaternion half-space of degree n, Manuscripta Math. 77 (1992) 215–235.
[19] Y. Kitaoka, A space of Siegel modular forms closed under the action of Hecke operators, Proc. Japan Acad. Ser. A Math. Sci. 70 (1994) 194–197.
[20] W. Kohnen, Lifting modular forms of half-integral weight to Siegel modular forms of even genus, Math. Ann. 322 (2002) 787–809.
[21] W. Kohnen, Linear relations between Fourier coefficients of special Siegel modular forms, Nagoya Math. J. 173 (2004) 153–161.
[22] A. Krieg, Modular Forms on Half-Spaces of Quaternions, Springer Lecture Notes in Math., vol. 1143, 1980.
[23] A. Krieg, The Maaß-space on the half-space of quaternions of degree 2, Math. Ann. 276 (1987) 675–686.
[24] A. Krieg, The Maaß-space and Hecke operators, Math. Z. 204 (1990) 527–550.
[25] S. Kudla, Splitting metaplectic covers of dual reductive pairs, Israel J. Math. 87 (1994) 361–401.
[26] J.-S. Li, On the classification of irreducible low rank unitary representations of classical groups, Compos. Math. 71 (1989) 29–48.
[27] C. Moeglin, M.-F. Vignera, J.-L. Waldspurger, Correspondence de Howe sur un corps p-adique, Springer Lecture Notes in Math., vol. 1291, 1987.
[28] I. Piatetski-Shapiro, S. Rallis, L-functions for classical groups, in: Springer Lecture Notes in Math., vol. 1254, 1987, pp. 1–52.
[29] G. Shimura, Confluent hypergeometric functions on tube domains, Math. Ann. 260 (1982) 269–302.
[30] G. Shimura, Euler Products and Eisenstein Series, CBMS Reg. Conf. Ser. Math., vol. 93, Amer. Math. Soc., 1997.
[31] G. Shimura, Some exact formulas on quaternion unitary groups, J. Reine Angew. Math. 509 (1999) 67–102.
[32] G. Shimura, Classification, construction, and similitudes of quadratic forms, Amer. J. Math. 128 (2006) 1521–1552.
[33] G. Shimura, Arithmetic and Analytic Theories of Quadratic Forms and Clifford Groups, Math. Surveys Monogr., vol. 109, Amer. Math. Soc., 2004.
[34] T. Sugano, Jacobi forms and theta lifting, Comment. Math. Univ. St. Pauli 44 (1995) 1–58.
[35] A. Weil, Basic Number Theory, third edition, Springer, 1974.
[36] S. Yamana, Determination of holomorphic modular forms by primitive Fourier coefficients, Math. Ann. 344 (2009) 853–862.
[37] S. Yamana, Degenerate principal series representations for quaternionic unitary groups, Israel J. Math., in press.
[38] D. Zagier, Sur la conjecture de Saito–Kurokawa (d'aprés H. Maass), in: M.-J. Bertin (Ed.), Sém. Deligne–Pisot–Poitou Paris 1979/1980, in: Progr. Math., vol. 12, Birkhäuser, Boston, 1981, pp. 371–394.